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XXIII.—*The Change in Thermal Conductivity of Metals on Fusion.*

By Alfred W. Porter, *D.Sc., F.R.S., and* F. Simeon, *B.Sc., Research Scholar University College, London.*

Received March 24, 1915.

It is well known that when fusion of a pure metal takes place there is, in most cases, a sudden drop in its electrical conductivity. For example, the electrical conductivity of liquid mercury is only about one-quarter of that of solid mercury at the same temperature. Exceptions are bismuth and gallium, for which the conductivity in the liquid state is double that in the solid. The object of the present investigation was to find whether there is a corresponding change in the value of the thermal conductivity. According to the simple form of the electronic theory usually given, the ratio of the two types of conductivity should be $\frac{4}{3}\left(\frac{\alpha}{e}\right)^2 \mathrm{T}$, where α and e are universal constants ; and although we know that this law is obeyed only roughly, yet the correspondence is sufficiently near to show that the two phenomena are intimately connected.

The outcome of the investigation is to show that the change in the thermal conductivity on fusion is of the same order as that of the electrical.

Outline of Method.

The method of measurement consisted in the determination of the gradient of temperature in a column of the metal contained in a glass tube, and heated at one end to such an extent that half the column is molten, while the cooler half is solid. The metals employed were sodium and mercury. In both cases the column was placed vertically. In order to diminish convective loss of heat from the surface it was heated at the top in the case of sodium whose melting point is above atmospheric temperature, while in the case of mercury the lower half was frozen by insertion of the end of the containing tube in solid carbon dioxide. The apparatus used in the case of sodium is shown in Fig. 1. The heating vessel contained lead which could be melted by means of a flame and kept at a constant temperature. Into the base of the vessel was screwed a piece of brass rod about $1\frac{1}{2}$ in. long and $\frac{3}{4}$ in. diameter. This brass rod served to convey the heat to the sodium with which it came into contact. The glass tube which contained the

sodium was furnished with 12 tubular depressions which were made by locally melting the glass with a fine flame, and then pressing it in with a knitting needle. These tubular depressions were destined to receive the thermoelectric junctions by means of which the temperatures at the various points of the column of metal could be determined. Each junction consisted

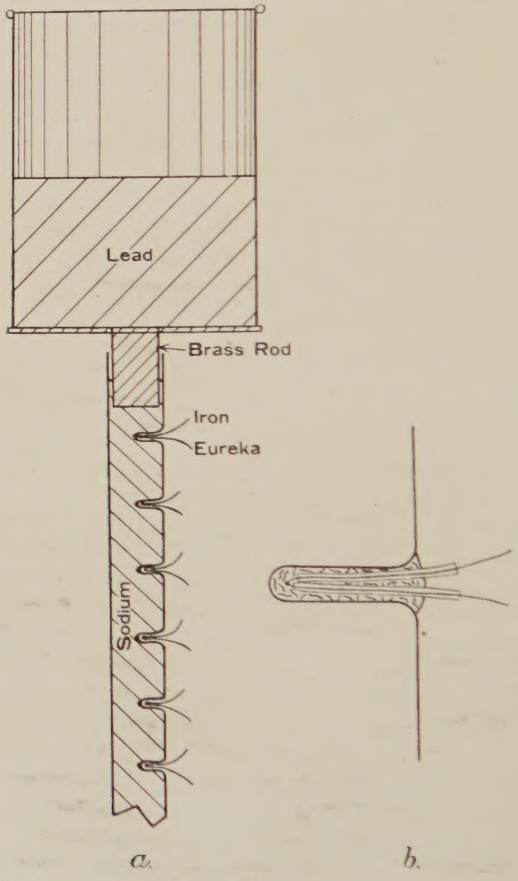

Fig. 1.

of fine iron and eureka wires (gauge 30-36). For the purposes of insulation fine glass tubes about $1\frac{1}{2}$ in. long were slipped over the wires to separate them one from the other, and were then fused to the wires. Each junction so formed was inserted in a tubular depression, and the cavity was filled up with a packing

of tinfoil so that the presence of the junctions should make as little disturbance as possible in the thermal stream lines in the sodium (Fig. 1, *b*). The arrangement is shown in Fig. 1, *a* and *b*.

The wires were all of the same length, and were connected by copper wires (all of the same length as one another) to a moving-coil galvanometer of low resistance in series with a resistance-box for altering the sensitiveness. By making the

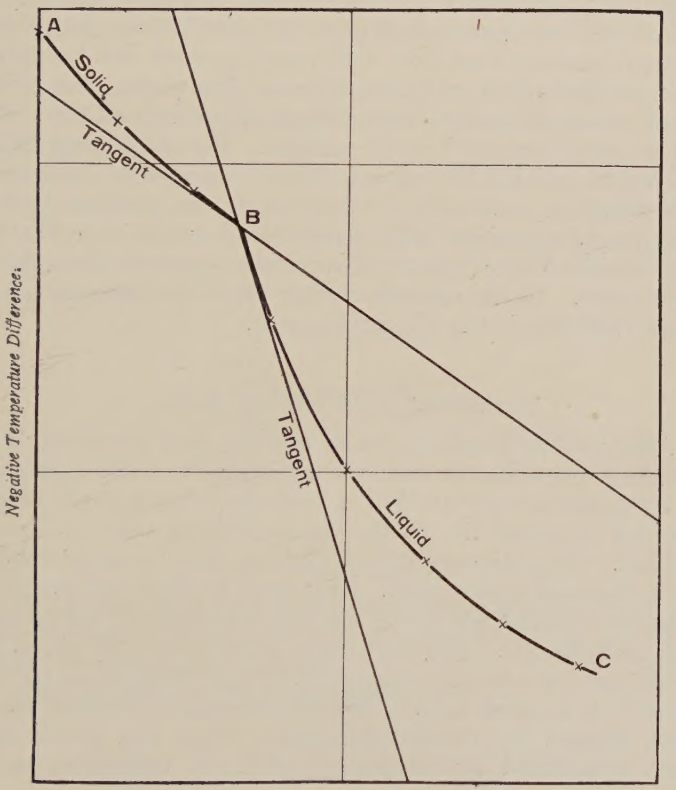

FIG. 2.—TEMPERATURE CURVES AND TANGENTS FOR MERCURY.

wires of equal length the temperature value of a scale division was made the same for all the couples. Since there was a resistance of 700 ohms in series with the galvanometer, any fine adjustment of these lengths was unnecessary.

The preparation of the sodium specimens was somewhat complicated by its solidity at ordinary temperatures, and its

oxidation on exposure to air. The following method was therefore adopted in preparing a bar. A glass tube, about 160 cms. long and 2 cms. diameter, was taken, and inset tubes made in its side as described above. About 10 cms. from the first of these tubes and 30 cms. from the nearer end, a constriction was made so that the end piece formed a funnel. A piston formed from another glass tube worked in from the other end. With the piston pushed well in, the tube was filled to a point above the constriction with pure dry paraffin oil. Pieces of sodium, carefully freed from oxide, were inserted, and the tube carefully heated to melt the sodium. The piston was then slowly drawn down, and more sodium added until all the inset tubes were covered by the sodium. The tube was then allowed to cool and the top cut off just below the constriction. The sodium rod so formed, in its casing of glass, was then ready to be placed in position below the lead bath, as shown in Fig. 1. The thermoelectric junctions were then packed into their inset tubes. In the case of mercury there was, of course, no special difficulty in the filling of the tube.

Description of Observations Made.

After putting a sodium rod in position, and connecting up the thermo-couples, the lead bath was heated to and maintained at a temperature (about 350°C.), such that when a steady state was set up about five of the thermo-couples were in liquid sodium. The galvanometer deflections due to each couple were then read in succession. This could be done very quickly, since the change from one to the next was made by a mercury cup interchanger. The cups were arranged in the arc of a circle, and each could be connected in turn with the centre by means of a radial arm. The readings thus obtained were then reduced by a calibration curve which had previously been determined by comparison of direct deflections with potentiometer readings.

These reduced readings were then plotted against the positions of the rod to which they correspond. To find from this curve the ratio of the conductivities of the liquid and solid, it is only necessary to find the two tangents which can be drawn at the point corresponding to the junction between the liquid and solid portions of the rod; because this ratio is inversely proportional to the ratio of the slopes of the two tangents.

In the case of mercury, arrangements had to be made to cool at the bottom instead of to heat at the top. The lower end was inserted into a carbonic acid snow and alcohol mixture. The whole tube was surrounded by a celluloid cylinder through which the thermo-electric couple wires were threaded. The object of this cylinder was to diminish the deposit of hoar frost in the lower part of the tube, which was otherwise very disturbing. The celluloid cylinder limited the rate of supply of fresh moist air to the tube, and thus achieved the desired object.

The change of slope is very much greater for mercury than for sodium. A typical curve for the former is shown in Fig. 2. The tangents at the point of change are also shown in this diagram. There is, of course, some difficulty in drawing in these tangents; hence, in different experiments considerable variation in slope was obtained. The following is a complete set of values for the ratio of the conductivity for the solid to that for the liquid at the same temperature :—

K_S/K_L for mercury.
4·44
4·62 (Curve on Fig. 2.)
3·28
3·76
3·83
3·53

3·91 = Mean K_S/K_L for mercury.

In the case of sodium it was found more satisfactory to plot $\log \theta$ against x, because the slopes, especially that on the liquid side, were then more constant.

The following slopes were obtained :—

Sodium.

Solid.		Liquid.
0·0279	.....	0·0230
0·0267	.....	0·0188
0·0248	.....	0·01885
0·0267	.....	0·0209
0·0269	.....	0·0200
Sum 0·1330		0·10155

Ratio of slopes $\frac{0\cdot1330}{0\cdot10155} = 1\cdot31 =$ mean K_S/K_L for sodium.

Electrical Conductivity.

The change of electrical conductivity with fusion had been determined by other observers both for mercury and sodium. The following determination for mercury are on record :—

Cailletet and Bouty (1885)	4·08
Dewar and Fleming (1896)	4·04
C. L. Weber (1885).....	4·10

In the case of sodium, previous determination of the change in electrical conductivity are :—

Matthiesson (1857)....	1·37—1·70
Bernini (1903).....	1·35
Northrup (1911).....	1·434 (Estimated from diagram.)

It was deemed advisable to make a fresh determination for sodium. A glass tube 0·5 cm. in diameter with two platinum wires sealed in at points about 5 cms. apart, was filled with sodium in a similar way to that described above. The tube was then set up in a horizontal electric furnace in such a way that the two platinum wires were near the centre of the furnace. Current was passed through the sodium, contact being made with it by stout copper electrodes. The potential-difference tapped off by the platinum wires was balanced on a potentiometer. No absolute measurements were made. Readings of the potential drop were taken at various temperatures below and above the melting point of sodium, and for both ascending and descending temperatures. The constancy of current through the sodium was checked by taking the potential drop along a piece of thick copper wire connected in series with it. The potentiometer readings were plotted against the temperature, and the ratio of the temperature coefficients for liquid and solid obtained by determining the slopes of the curves on the two sides of the melting point.

Four independent values obtained for the ratio were—

1·453
1·402
1·430
1·467

giving a mean of 1·438. This value is very similar to Northrup's, but is rather higher than our thermal values.

It may be of interest to mention some of the other changes which occur during fusion. There is, of course, a change in the specific energy (or internal latent heat)—that of the liquid being

the greater; a change in the density that of the liquid being less in the case both of sodium and mercury, but greater in the case of bismuth; a change in the coefficient of expansion that of the liquid being the greater; the specific heat of sodium undergoes no appreciable change, but its rate of variation with temperature changes from positive to negative on fusion (Ezer Griffiths, Roy. Soc. "Proc.," A., Vol. LXXXIX., 1913-14, p. 561); the temperature coefficient of the electrical resistance of mercury changes from 0·00455 between —55°C. and —40°C. to 0·000834 between 0°C. and 5°C.

With regard to thermo-electric effects the evidence is somewhat more conflicting. W. B. Burnie ("Phil. Mag.," 1897) finds a continuity in the curve of E.M.F. against temperature in the case of a copper-mercury element with, however, a marked change in its slope, $\frac{dE}{d\tau}$, at the melting point of mercury. The slope changes abruptly from a large nearly constant positive to a smaller negative value. This would, of course, imply a change in the Peltier coefficient $\left(\tau\frac{dE}{d\tau}\right)$. Assuming that the Peltier coefficient is proportional to the logarithm of the ratio of the concentrations of the electrons in copper and mercury, his values would give $n_{\text{solid}}/n_{\text{liquid}}$=about 5. This is sufficiently near to the ratio found for the electrical conductivities to tempt one to suppose that the two phenomena are telling consistent stories. On the other hand, Peddie and Shand ("Proc." Roy. Soc., Edin., 23, p. 15, 1900) find no change; nor also did P. Cermak ("Ann. der Phys.," 26, 1908, p. 521) for either the thermo-electric power or the Peltier effect.

Lastly, the photo-electric effect shows no change on fusion of the metal (sodium) according to Dember ("Ann. der Phys.," 23, 1907, p. 957). These results are discussed by E. Wagner ("Ann. der Phys.," 33, 1910, p. 1484).

The experimental part of this Paper was finished in July. 1913. In September, 1913, of the "Physical Review" appeared a Paper by J. W. Hornbeck on "Thermal and Electrical Conductivities of the Alkali Metals," in which is described a very similar method of investigation. In the case neither of sodium nor potassium was the investigation carried as high as the melting point; the thermal conductivity for both the solid and the liquid was determined for potassium-sodium alloy, the value being less in the liquid state. As the values were not obtained at the melting point itself, and as

there is only one value for the solid, it is not possible to state the exact ratio of reduction for this case.

The only other determination with which we are acquainted is one made by C. Barus ("Phil. Mag.," 33, p. 431 (1892), on thymol, a dielectric). He finds for the ratio of the thermal conductivity of solid to that of liquid the value 359/313. He had previously measured the change in specific heat. The corresponding change in thermometric conductivity is 1077/691.

ABSTRACT.

The change in question was determined for mercury and for sodium by finding the temperatures at different points of a cylinder of the metal contained in a glass tube. The ends of the cylinder were maintained at such temperatures that the metal was liquid half-way down its length, the remaining part being solid. The temperatures were taken by means of thermoelectric junctions inserted in narrow tubular depressions which had been formed in the glass tube by forcing a knitting needle down into the locally heated glass. The ratio of the thermal conductivity for solid and liquid was estimated from the slope of tangents drawn to the temperature-curve on each side of the melting point. The values of these ratios are of the same order as the ratio of the corresponding values of the electrical conductivities. The mean value for mercury is 3·91, and for sodium 1·31.

A summary list is given of other data concerned with fusion.

DISCUSSION.

Prof. O. W. Richardson congratulated the authors on their satisfactory treatment of an important problem. The change of thermal conductivity with fusion had not, he believed, been attacked experimentally before. In listening to the reading of the Paper one felt impressed by the small amount of effort and the simplicity of the apparatus with which the authors had been able to solve what was generally regarded as a very difficult problem. This research, it was interesting to note, was one of the few of which the results agreed with the deductions from the elementary electronic theory.

Prof. F. G. Donnan referred to the researches carried out at his suggestion by Dr. C. M. Stubbs on the radiation emitted by copper, silver and gold at high temperatures. These experiments showed that there was a marked change in the distribution of energy throughout the visible spectrum when passing from solid to molten metal, and pointed to some change in the electronic structure, or in the inter-electronic forces, occurring at the transition from solid to liquid. The complexity of the problem had, so far, prevented any mathematical treatment, but there appeared no doubt that results such as these, coupled with the discontinuities in thermal and electrical conductivity, would be of value in throwing light on the inner mechanism of the phenomenon of melting. Photometric measurements of the light radiated by polished plates of metals at high temperatures presented considerable difficulties owing to the tendency of the "flowed" surfaces to crystallise, but in spite of these difficulties Dr. Stubbs had clearly demonstrated the existence of an

abrupt change in the radiation-spectrum on passing from solid to liquid in the case of certain metals.

Dr. PORTER, in reply, said that while isolated measurements of thermal conductivity of some substances above and below the melting point had been made by previous workers, in no case that he was aware of had sufficient data been obtained from which to deduce the sudden change which takes place on fusion. He was cognisant of the work of Dr. Stubbs but had always thought the nature of the surface was of such importance in radiation experiments as to make it difficult, on account of crystallisation and contamination of the surface at high temperatures, to obtain conclusive results. He had forgotten at the moment the degree of consistency obtained by Dr. Stubbs, but if the results were more or less in agreement they would undoubtedly point to a change at the fusion point in the radiation emitted by the substances.

XXIV. *An Instrument for the Optical Delineation and Projection of Physical Curves.* By J. A. Fleming, *M.A., D.Sc., F.R.S.*

Received March 6, 1915.

It is often desirable and necessary to exhibit a physical curve such as a characteristic curve, hysteresis, or resonance curve in process of delineation, and to be able to record it by photography. The author has devised an apparatus which does this very conveniently, and can be applied in the delineation of physical curves of many kinds.

It consists of a base board on which is placed on suitable supports a rectangular rocking mirror pivotted on an axis parallel to one of the long sides of the rectangle. This mirror may be 1½ in. wide and 3 in. or 4 in. long. To the axis of the mirror is affixed a lever by which it can be turned through a certain limited angle. This lever is restrained by a spiral spring and pulled by a string which passes over a pulley. When the string is pulled the mirror is tilted about a horizontal central axis, and when the string is released the mirror returns to its original position. Over this mirror is another much larger one which is slung on and rotates round an axis at right angles to that of the first-named mirror, and serves to project on to a screen a ray of light reflected from the tilting mirror (*see* Fig. 1). On the same base board, which carries the supports of these mirrors, is fixed a thin wooden hoop like the rim of a wheel about 18 in. in diameter, 0·5 in. wide and 0·5 in. deep. This hoop is wound over with one layer of silk-covered Eureka, or high-resistance wire, in closely adjacent turns. The silk is rubbed off the top surface so as to expose the bare wire. This winding surrounds about three-quarters of the perimeter of the hoop. It is formed of No. 22 Eureka wire, and measures in resistance about 40 ohms. Three terminals are attached to it, one at the middle and one at each end. This wire forms a potentiometer wire, and a battery of 1 to 10 small secondary cells has its poles attached to the ends of this wire so as to make a fall of potential down it. In the centre of the hoop there is a pivot and a vertical axis with a radial metal arm, the end of which presses on, and moves over the bare exposed turns of wire on the hoop. A terminal is in connection with this axis. It will be seen that if the radial arm is moved either way from the central position in which it makes contact with the middle point of the potentiometer wire it takes off a certain

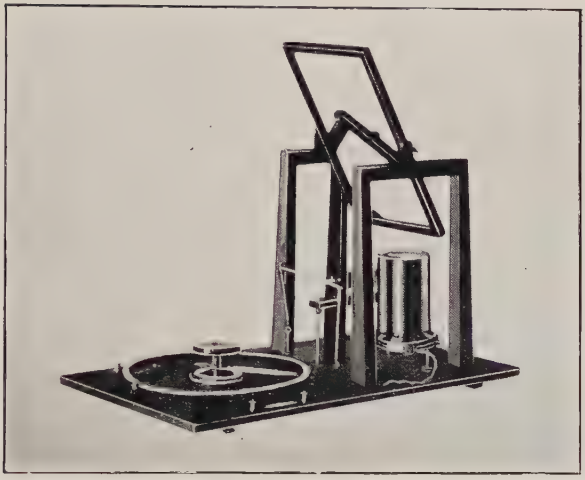

Fig. 1.—View of the Curve Tracer, showing Projection Mirror and Circular Potentiometer.

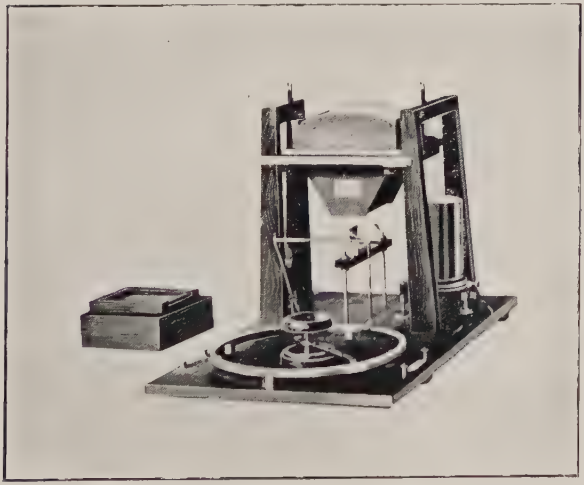

Fig. 2.—View of the Curve Tracer, showing Tracing Desk and Photographic Camera Attachment.

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fraction of the voltage of the battery, and we can create between the middle terminal of the potentiometer wire and the terminal in connection with the revolving arm any required fraction of the half battery voltage either in one direction or in the other. On the board is also placed a mirror galvanometer of movable coil or movable needle type or else a mirror magnetometer. A thin parallel beam of light is sent out from an electric arc lantern, and falls on the mirror of this galvanometer or magnetometer, and is thence reflected to the tilting mirror, and from that to the inclined projection mirror, and thence to a screen. The mirror galvanometer is so placed that right or left deflections of its coil or needle cause the ray to travel along the tilting mirror parallel to its long axis, and, therefore, the spot of light falling on the screen moves vertically in response to deflections of the galvanometer or magnetometer.

The thread attached to the lever fixed on the shaft of the tilting mirror is wound round a pulley on the shaft or axis of the radial arm, and so adjusted that when the radial arm is rotated the mirror is tilted through a corresponding angle. Hence the spot of light projected on to the screen has two motions—one, a horizontal motion which is proportional to the angle through which the radial arm is turned, which again is proportional to the P.D. created between the middle terminal of the potentiometer wire and the terminal in connection with the radial arm. Again, the spot of light can have a vertical motion imparted to it which is proportional to the deflection of the galvanometer coil or magnetometer needle, which, in turn, is proportional to the current through the galvanometer or to the deflecting magnetic field, which is at right angles to the direction of the needle itself.

Hence, if we cause the spot of light to be actuated by these two motions at once, it will describe a curve which will represent the relation between two variables, one proportional to the above mentioned voltage, and the other to the above mentioned current or field. Instead of receiving the reflected ray of light upon a screen it may be passed through a cylindrical lens placed over, and with axis parallel to, the tilting mirror, and may then be allowed to fall upon a photographic plate in a plate holder and record on it the curve in question (*see* Fig. 2).

As an example of one simple use of the above instrument, we may explain how it can be used to delineate and photograph a

magnetic hysteresis curve for iron. For this purpose a coil or helix is required wound with many turns of fine wire. The one used by the author is a coil wound with silk covered No. 36 S.W.G. copper wire in 37,363 turns and 21 layers. The wire is coiled on a brass tube 45·7 cm. long between the cheeks, and has a resistance of 979·3 ohms at 15° C. The value of the magnetic field per milliampere in the interior, near the centre, is approximately

$$H=\frac{4\pi}{10}\frac{N}{L}\frac{1}{1,000}=1{\cdot}021 \quad , \quad . \quad . \quad . \quad . \quad (1)$$

and the magnetic field per volt applied to the coil terminals is

$$H=\frac{4\pi}{10}\frac{N}{LR}=1{\cdot}05. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, the field is nearly 1 C.G.S. unit per volt impressed at terminals of the coil. Accordingly, by using a battery of 10 cells on the circular potentiometer wire and applying the tapped off voltage to the terminals of this coil we can create any magnetic force between —10 and +10 in the interior of the helix. If, then, we place on the curve tracer a mirror magnetometer and control the needle so that it stands with its axis parallel to the axis of the tilting mirror, and if we place the above-mentioned magnetising coil at some little distance away with its axis perpendicular to that of the magnetic needle and put in the helix an iron wire of length at least 200, or, better still, 400 times its diameter, we can apply to this wire a known magnetising force. It will then have created in it a certain magnetisation, and this will create an external magnetic field which will deflect the magnetometer needle through an angle which, if small, is proportional to the magnetisation and nearly to the flux density at the centre of the wire. Hence, if a ray of light is thrown on the mirror of the magnetometer, and reflected from the two mirrors of the curve tracer on to a screen we shall be able, by rotating the radial arm from one position on the potentiometer wire to another similar one on the other side of the middle point and then back again, to carry the wire through a magnetic cycle, and cause the spot of light on the screen to delineate a hysteresis curve or loop for any required range of magnetising force H between, say, +10 and —10. Moreover, we can calibrate the instrument so as to interpret the scale of the resulting curve and, therefore, calculate the energy dissipated per cycle of magnetisation (*see* Figs. 3 and 4).

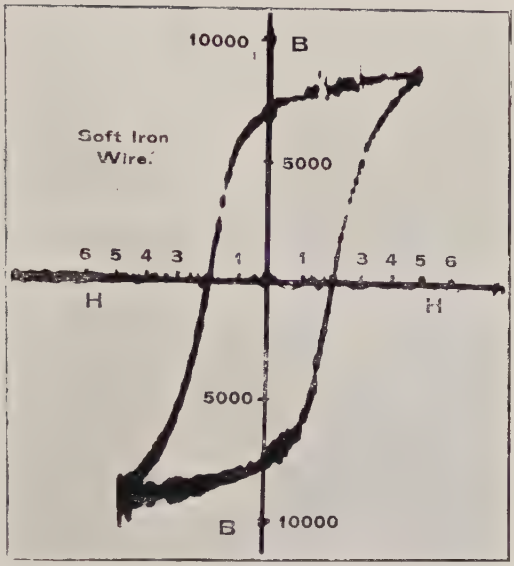


FIG. 3.—PHOTOGRAPH OF MAGNETIC HYSTERESIS CURVE OF AN IRON WIRE, TAKEN WITH THE CURVE TRACER.

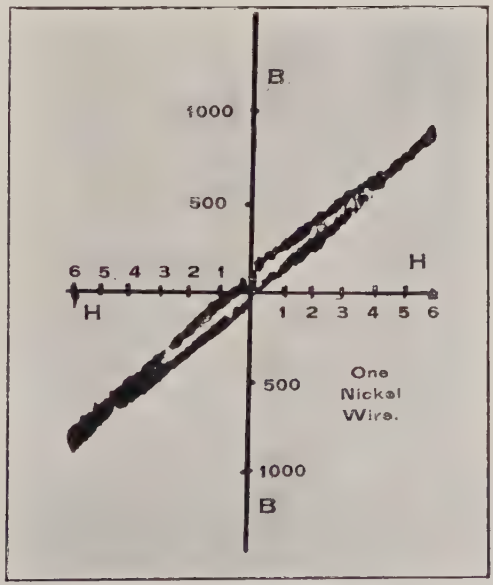


FIG. 4.—PHOTOGRAPH OF THE MAGNETIC HYSTERESIS LOOP OF NICKEL WIRE.

To face page 318.

To do this we replace in the exact position of the iron wire another helix or solenoid of equal length and known number of turns per centimetre, through which we can pass a measured current. Let this helix have N_1 turns per centimetre and let a current, A_1, be passed through it, which just makes the same magnetometer deflection as the iron wire at the extreme or maximum magnetisation. Then, $1{\cdot}25\,N_1A_1$ is the flux or field over the greater portion of the interior of the solenoid, and this must, therefore, be the same as the total flux in the iron-wire under the above conditions. Therefore, knowing the cross-section of the iron, we can find the flux density B in it, and hence the flux density scale for the hysteresis loop. Knowing the scale of the ordinates, we can then find the area of the hysteresis loop by marking out the path of the spot of light as it travels round one complete magnetic cycle.

If this path is marked off on paper we can cut out this curve and weigh the paper. We can then cut out of the same piece of paper a square of area known in units selected for B and H, and weigh that piece, and from the ratio of the weights find the area of the hysteresis loop in terms of that unit of area which is equal to the product of the lengths on the two axes which stand respectively for the unit of magnetising force H and the unit of magnetic flux density B. We know, then, that the value of $\frac{1}{4\pi}\int H\,dB$, or nearly 7/88ths of the area of the loop in this unit, gives us the work done in ergs in carrying unit volume of the iron round one complete magnetic cycle. Hence, we obtain the hysteresis for the maximum flux density employed, and, therefore, the hysteretic constant of the iron generally.

The calibrating coil employed by the author consisted of No. 22 silk-covered copper wire wound on a brass wire in 4,687 turns of eight layers. The length of the coil was 48·0 cm., and the diameter of the brass wire was 0·325 cm. diameter. Hence, the area of cross-section of the wire is 8·3 mm. square. The resistance of the coil was 5·73 ohms at 15°C. Hence, the magnetic force H in the centre is 122·5 units per ampere, or 21·4 per volt on the terminals. The cross-sectional area of the iron wires used in the above hysteresis measurement was 2·075 $(\text{mm.})^2$, and hence the cross-section of the iron was 0·25 of the cross-section of the calibrating coil. The total flux embraced by the turns of this calibrating coil per ampere of current through it can be found by a single experiment with an embracing secondary coil and a ballistic galvanometer.

A second illustration of the use to which this instrument can be put is in the delineation of resonance curves.

In this case we replace the circular potentiometer wire by a variable air condenser of the multiple semicircular plate type, the capacity of which can be varied by rotating the insulated milled head which moves one set of plates. In the case of such a condenser the effective capacity is very nearly proportional to the angle through which the movable plates are turned, and can be made exactly by the use of a Seibt air condenser.

The string attached to the lever on the axis of the tilting mirror is then wound round the shaft of this condenser and adjusted so that rotating the movable plates pulls over the tilting mirror from one extreme limit of movement to the other. A suitable mirror galvanometer of low resistance is placed in position and connected to a thermocouple, pressing on a short length of fine high-resistance wire, which is inserted in the circuit of the variable condenser, and the said circuit is completed by a coil of suitable inductance, L, so that the oscillation constant of the circuit ($= \sqrt{CL}$) has a value near to that of the circuit to be tested. Let it be then desired to obtain the resonance curve of a wireless telegraph spark transmitter or of any other oscillation circuit. The above described condenser circuit is loosely coupled with the circuit under test, and a ray of light from an arc allowed to fall on the galvanometer mirror and to be reflected thence to the tilting mirror and projection mirror and then on to a screen. It is obvious that if the rotating handle or head of the variable condenser is slowly turned from one extreme position to the other, the circuit will gradually be brought up into resonance with the circuit under test, and then pass beyond it. The spot of light on the screen will, therefore, be caused to move horizontally by the change in capacity by a distance proportional to that change, and vertically by a distance which is approximately proportional to the square of the current in the condenser circuit. Hence, the spot describes the well-known path of a resonance curve which is more or less peaked in proportion as the total decrement of the two circuits is less. A few simple measurements suffice to calibrate or show the scale of the horizontal and vertical movements.

Taking, then, the well-known modification of Bjerknes' decrement formula applicable in this case, and calling C_r the capacity of the condenser setting corresponding to resonance,

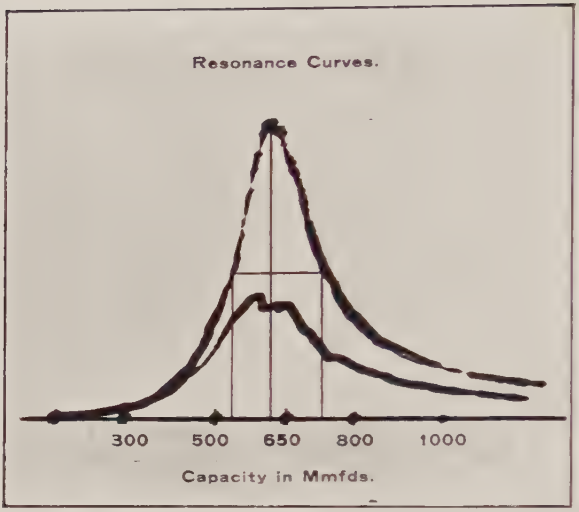


Fig. 5.—Resonance Curves photographed with Curve Tracer.

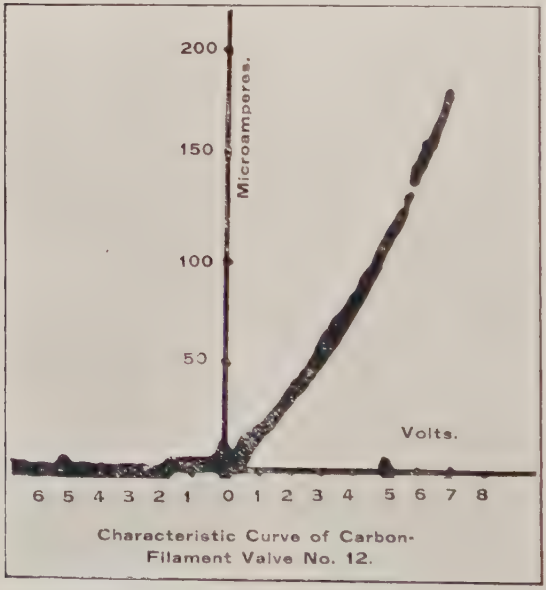


Fig. 6.—Photograph of Characteristic Curve of Fleming Oscillation Valve.

To face page 321.]

and I_r the resonance or maximum current in the condenser circuit, whilst C and I stand for the capacity and current corresponding to any other setting, we have, provided $(C_r—C)/C_r$ is a quantity not exceeding, say, 5 per cent., the expression

$$D=\frac{\pi}{C_r}(C_r—C)\sqrt{\frac{I^2}{I_r^2—I^2}}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where D is the sum of the decrements per complete period of the circuit under test and of the circuit of the curve tracer. Now, if we take such a value of the capacity C as to make the current

$$I=\frac{1}{\sqrt{2}}I_r=0{\cdot}707\,I_r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

then the quantity under the $\sqrt{}$ in the above formula (3) becomes unity, and we have

$$D=\frac{\pi}{C_r}(C_r—C). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We can once for all find the value of C_r from the calibration curve of the condenser, and therefore the value of π/C_r, and therefore it is clear that the total decrement is proportional to the horizontal shift of the spot of light from the zero position which corresponds to a reduction in the maximum value of the mean-square current to half its value. Since the ordinates of the resonance curve will be nearly proportional to the mean-square current we have merely to ascertain the horizontal shift of the spot of light from its position corresponding to the maximum ordinate of the resonance curve when that ordinate is reduced to half its value.

It is quite easy by known methods to find once for all the decrement of the condenser circuit of the curve tracer, and, therefore, by subtraction, to find the decrement of the oscillation circuit under test.

The resonance curve can be photographed by projecting the spot of light on a slow photographic plate or on Eastman's bromide paper, by causing it to move slowly over its path once or twice. We have, therefore, in this appliance a very simple method of quickly taking a resonance curve and recording it (*see* Fig. 5).

By inserting in one or other of the oscillation circuits a piece of high-resistance wire, we can show how much the sharpness

or the relation of pressure and volume, expansion and temperature in cyclical operations.

In magnetic investigations it is sure to prove useful, because we can operate with the magnetised wire under test not only in a horizontal position, but in a vertical one. We can thus investigate the effect of vibration, torsion, temperature and electric oscillations on the hysteresis of iron and other magnetic metals, and see in a moment whether any particular physical or electrical operation increases or diminishes the magnetic hysteresis (*see* Fig. 8). The fact that the magnetic samples are used in the form of wires or narrow strips cut off a sheet is a great advantage and very much more simple than when rings have to be employed. If the wires employed are about 1 mm. in diameter and 400 mm. long they are in effect magnetically equivalent to rings. We can use this curve tracer to investigate in this manner samples cut from the actual sheet steel to be employed in making the core of a transformer or dynamo armature.

It is also easy to use the instrument to exhibit the recalescence points of iron by heating a wire by an electric current to bright redness, and then allowing it to cool, and causing the temperature to be measured by a thermo-electric junction, whilst the tilting mirror is slowly moved over by a clock or mechanism which records the time. The instrument exhibited has been made in the Pender Electrical Laboratory by my assistants, Messrs. P. R. Coursey and Williment, and Mr. Coursey has rendered me skilful assistance in taking the photographs exhibited of various physical curves.

Note Added April 26, 1915.

The possibilities of the above described instrument must not be altogether judged by the illustrations given in the above Paper from photographs taken with it. In the instrument actually constructed the optical arrangements were by no means as good as they can be made. By the use of properly worked galvanometer or magnetometer mirrors, and the adoption of well-known means for securing a smaller well-defined spot of light on the plate, the photographic curve could be made much sharper and uniform, and the definition far better.

Also, the author has, since the reading of the Paper designed another arrangement for the production of a uniformly varying

current by the rotation of an axis as follows : A ring is formed of high-resistance alloy which should be in the form of a strip or wire capable of carrying a current of 30-50 amperes. This ring has two electrodes attached to it at opposite ends of a diameter so that a current may be passed through the two halves in parallel by means of one or more large storage cells. A bar of wood or other insulating material is pivoted on an axis which is fixed at the centre of the ring and this axis has one end of the string attached to the tilting mirror fastened to it. The axis must be of such a diameter that when turned through one semi-revolution the string pulls over the tilting mirror through the entire range of motion permissible. The rotating diametral arm carries a pair of brushes or rubbing contacts which press on the two opposite sides of the ring, and these brushes are connected by flexible connectors or by rubbing contacts with two fixed terminals to which are attached the ends of the circuit in which it is desired to produce a continuously varied current in one or other direction.

It will be evident that when the diametral arm contacts are turned so that they make contact with two points on the ring 90 deg. removed from the two electrodes of the ring, then no current flows through the above circuit. If, however, the diametral arm is turned so that it connects the two electrodes of the ring then the current in the circuit is a maximum one way or the other. Hence by slowly moving this arm through 180 deg. and back we can carry the current in the circuit through a complete cycle and if this circuit includes a magnetising coil we can apply to an iron core within it, a cyclical magnetising force. This ring and its diametral arm would then take the place of the circular potentiometer in those cases in which it is desired to make the horizontal movement of the spot of light represent a larger current than can be produced by the circular potentiometer. The other arrangements would remain the same. It will, however, in general be found most convenient to retain the potentiometer method described in the Paper.

In the above described current-producing arrangement, there is, of course, a considerable waste of current which flows round the two sides of the ring and is generally large compared with that which flows through the diametral circuit. If the radius of the circular ring of resistance wire is denoted by r, and if ρ is the resistance per unit of length of this wire, and if R is the resistance of the radial arm and any coil included in it,

then it is easy to show that the current flowing through this radial arm is

$$x=\frac{V(\pi-2\theta)}{\pi R+2\theta(\pi-\theta)r},$$

where θ is the angular distance of the radial arm from the two diametral points on the ring at which the current enters and leaves, and V is their P.D. For if $\theta=0$ then $x=V/R$, and if $\theta=\pi/2$ then $x=0$, as it should be. Also if R is large then x is nearly proportional to $\pi/2-\theta$.

In the discussion which followed the exhibition of this instrument to the Physical Society, Prof. Silvanus Thompson asked if the instrument had received a name. Acting on a suggestion kindly given by Prof. Arthur Platt the writer proposes as the name for this instrument the *Campograph*, from Καμπη, a curve.

ABSTRACT.

This instrument is designed for projecting on to a screen or photographing on a plate such curves as magnetic hysteresis, resonance, or characteristic curves which can be performed slowly, or are non-periodic or non-repetitive. It consists of a base board on which is fixed a thin wooden ring wound over with one layer in close turns of insulated Eureka wire. The insulation on the top surface is removed so as to expose the wire. At the centre of the ring there is a pivot which carries a radial arm the end of which rubs on the bare Eureka wire. The ends of the Eureka wire are connected to a battery of n cells and the middle point of the wire is connected to a terminal, and also the centre of the radial arm to another terminal. The arrangement forms a circular potentiometer so that any required fraction of the battery voltage can be produced in either direction between the last-mentioned two terminals. On the same base there is a rectangular mirror slung on a longitudinal axis, which mirror is tilted by a thread wound round the shaft of the radial arm of the potentiometer. Hence, if a ray of light from a mirror magnetometer or galvanometer is reflected to the tilting mirror and thence on to another fixed mirror with its axis at right angles to that of the other and thence to a screen, the spot of light will have two motions, one a horizontal one proportional to the displacement of the radial arm, and, therefore, to the P.D. of the two terminals, and another vertical motion proportional to the deflection of the galvanometer or magnetometer. If, then, we place behind the magnetometer a long coil having a long iron wire within it and send a current through this coil which is supplied from the potentiometer travelling terminals, the spot of light will be actuated by two motions, one a horizontal one proportional to the magnetising force, and the other a vertical one proportional to the central flux density in the iron. It will, therefore, describe a hysteresis curve when the radial arm is moved to and fro cyclically through a certain arc or angle. In the same manner the

instrument can be used to delineate characteristic curves of wireless detectors or rectifiers or other devices. By using a rotating condenser of variable capacity in place of the circular potentiometer, the instrument can be used for delineating resonance curves of wireless telegraph plants.

DISCUSSION.

Prof. S. P. Thompson complimented the author on the simplicity and effectiveness of his apparatus in which, as was characteristic of Prof. Fleming's devices, were embodied the results of much careful thought and painstaking experiment. He himself had often felt the desirability of having an arrangement by which one could vary the magnitude of an electric current in such a way that it should be directly proportional to some movement in a straight line. He had thought out various potentiometers in this connection, but the difficulty with potentiometers was the small current obtainable from them which was a disadvantage in magnetic work. Prof. Fleming had surmounted the difficulty of straight line proportionality by working round an axle instead.

Mr. W. Duddell joined in congratulating the author on the wide utility of his instrument. He recalled that when delivering his first lecture on the oscillograph he had been confronted with the problem of projecting the motion of a spot of light in two co-ordinates on a screen and had used a very similar arrangement to Prof. Fleming's. He also had a circular rheostat, but instead of a string connection to the mirror he had used a cam. This apparatus had worked very satisfactorily.

Prof. E. Wilson said that the apparatus was undoubtedly admirable for demonstration purposes, and one naturally wondered what degree of precision would be possible with it if applied to research. It appeared from the step and stair motion of the spot on the screen that the speed would be limited on account of the damping of the magnetometer or galvanometer employed.

Prof. S. W. J. Smith described a method which he had found to be simple and effective for tracing hysteresis loops. An ingenious rheostat had been devised by Prof. Morris in which by the rotation of a handle two contacts are made to slide in opposite directions along a resistance which is connected directly across the battery. The voltage at the contacts varies continuously between $+$ the whole P.D. of the battery, being zero when they are at the middle. This rheostat is mounted vertically, and a white card of suitable size is attached to one of the moving contacts from which the magnetising current is taken. Then if the magnetometer spot is focussed on the card and its position marked as the current is varied the hysteresis loop is easily traced out. The Morris resistance satisfies some at least of the requirements mentioned by Prof. Thompson, though for the method he had described simpler rheostats of similar type could also be used.

Prof. Morris agreed that the question of getting a current which is proportional to some linear motion was a difficult one. Before arriving at the design of the rheostat mentioned by Dr. Smith he had tried various other types, including a circular one similar to Dr. Fleming's. The main point about his rheostat was the magnitude of the currents which could be taken from it. It would stand 20 amperes for a short time.

Mr. A. F. Hallimond: As the author of the Paper mentioned by Prof. Fleming, he might perhaps explain very briefly the nature of the apparatus referred to. A slide which he exhibited showed a mirror galvanometer with one needle, which was freely suspended at the centre of the two coils, which were at right angles to one another. The beam of light from a small 4-volt lamp was focussed by a lens, and, after reflection from a thin glass plate, inclined at 45 deg. to the axis of the needle, travelled horizontally along the common diameter of the coils to the needle. A mirror attached to the needle re-

versed this beam, which came to a sharp focus on a screen at a distance of about 2 ft. Since writing this account his attention had been called to the curve tracer designed by Dr. Searle. In principle, he believed that the two pieces of apparatus were very similar. The assumption underlying Prof. Fleming's apparatus, that the voltage was proportional to the length of the potentiometer wire, was only accurate if the crystal had a very high resistance ; this difficulty was completely avoided by the use of the other type of apparatus.

Prof. FLEMING, in reply, thanked the various speakers for their remarks and said that he hoped to improve considerably on the apparatus especially as regards the optical arrangements. When this had been done he thought it would be extremely accurate and quite suitable for research work.

XXV. *The Stability of Some Liquid Films.* *By* P. Phillips, *D.Sc., and* J. Rose-Innes, *M.A.*

Received March 23, 1915.

Many teachers of physics must have felt that the literature at present published on the theory of surface tension does not meet their requirements when they are engaged in lecturing on the subject to students. Such a book as Poincaré's "Capillarité" assumes considerable attainments in the reader; while the so-called proofs in Maxwell's article in the "Encyclopædia Britannica" and in text-books which employ easier mathematics are often wanting in rigour. The present Paper is an attempt to supply this deficiency; to indicate, in fact, how we can prove some of the interesting theorems in the subject by methods at once simple, rigorous and not too clumsy. We may begin by considering the form of an equilibrium surface which is symmetrical about an axis. The best and best known investigation on the subject is probably that given by Maxwell in the article in the "Encyclopædia Britannica." Maxwell's logic on this point is unexceptionable, but pedagogically his work is marred by two minor blemishes: (1) He introduces the notion of a rolling curve abruptly without his previous work having in any way led up to the idea; and (2) he assumes more knowledge of conics than is necessary. We believe that a slight deviation from his proof enables us to get rid of these blemishes.

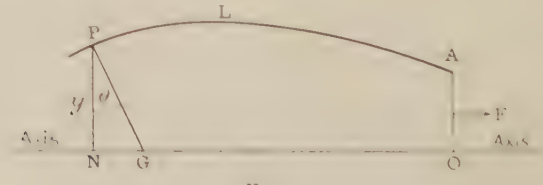


Fig. 1.

Consider the equilibrium of the portion of the film whose surface is produced by the rotation of ALP round the axis ON. AO is the radius of the end of the film which is kept stationary by having a force, F, applied to it parallel to the axis. PG is the normal to the surface and PN the normal to the axis, and

the angle NPG is θ. Then, if p is the excess of pressure inside the film over that outside, and T the surface tension,

$$F+p\pi y^2=2\pi y 2T\cos\theta,*$$

or

$$\text{Cos }\theta=\frac{p}{4T}\left(y+\frac{F}{p\pi y}\right).$$

To simplify the form of the equation write b^2 for $F/p\pi$ and a for $2T/p$. Then we obtain

$$\text{Cos }\theta=\frac{y+\frac{b^2}{y}}{2a}.$$

To obtain the form of the curve from this equation produce PN (Fig. 2) for a distance $NL=\frac{b^2}{y}$, and produce PG until it

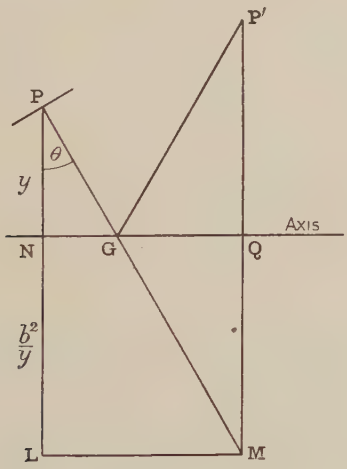


Fig. 2.

meets in M the line drawn through L parallel to the axis. Then PM is evidently equal to $2a$.

Draw MQP′ perpendicular to the axis making QP′ equal to QM, and join GP′. Then GP+GP′ is also equal to $2a$. Also the distance PP′ is given by

$$PP'^2=PM^2+MP'^2-2PM\,.\,MP'\cos\theta,$$
$$=4a^2-4b^2.$$

* This equation may also be derived from the second order differential equation, $P=2T\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$ by one integration.

Therefore, as y and ψ vary from point to point on the curve, G lies on an ellipse whose foci are P and P′, and whose major and minor axes are respectively $2a$ and $2b$. Since the two angles PGN and P′GQ are equal, the axis is a tangent to the ellipse, and since PG is a normal to the curve, the whole curve is described by P when the ellipse rolls on the axis.

The pressure within the film being $2T/a$, it depends on the major axis only of the ellipse.

If the pressure within the film is less than outside, p is negative, and it can be shown by a method similar to the above that the curve becomes the roulette of one focus of a hyperbola.

If the pressure inside the film is the same as outside, p is naturally zero and the equation becomes

$$y \cos \theta = 2a.$$

This can be shown just as simply to be the roulette of the focus of a parabola which rolls on the axis, the latus rectum of the parabola being $8a$.

By substituting dy/dx for $\tan \psi$ and integrating, and making $x=0$, when $\theta=0$, we get

$$\frac{y}{2a} = \cosh \frac{x}{2a},$$

or, putting c for $2a$,

$$\frac{y}{c} = \cosh \frac{x}{c}.$$

This is the well-known equation of the catenary with its directrix as the axis of x and the axis of symmetry as the axis of y. The equilibrium form is therefore a catenoid generated by the revolution of a catenary about its directrix.

The Stability of a Spherical Bubble.

Both Plateau and Maxwell seem to have been somewhat careless in stating the conditions of their films when they investigated the stability. The necessity of stating the conditions exactly is illustrated by means of the spherical soap bubble. A bubble of radius a and surface tension T will be in equilibrium with a source of gas kept at a constant pressure $4T/a$ above the atmospheric pressure. If the bubble expands, its equilibrium pressure* is reduced, and the source will force

* The term "equilibrium pressure" is used to denote the pressure with which the film would be in equilibrium, and the term "neighbouring equilibrium form" will be used to denote the neighbouring equilibrium form having the same equilibrium pressure.

more gas into it and expand it still further. Similarly any small shrinkage of the bubble will go on increasing. The bubble is thus always unstable at constant pressure.

For a completely closed bubble we must introduce Boyle's law as a condition. The equilibrium pressure to start with will be $4T/a$. If, therefore, the radius changes by a small amount to a_1, the equilibrium pressure becomes a/a_1 times as great, while the actual gas pressure becomes a^3/a_1^3 times as great. Thus, the change in gas pressure more than compensates for the change in equilibrium pressure, and for this type of displacement at any rate the bubble is stable.

The Stability of a Cylindrical Film.

This may also be considered under the same two conditions, first, at constant pressure, and, second, when the film encloses a constant mass of gas.

1. *At Constant Pressure.*—Let LM, NP represent the section of a cylindrical film whose axis is QR.

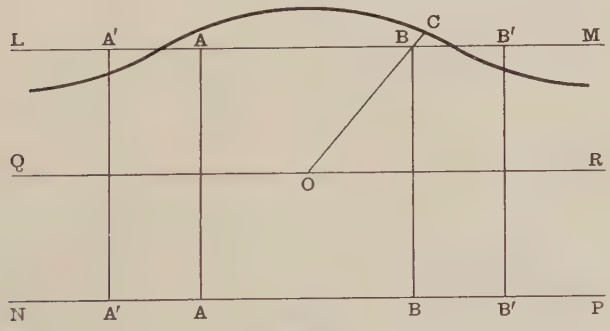


FIG. 3.

Then LM is the roulette of a focus of an ellipse which has become a circle—*i.e.*, of an ellipse which has zero eccentricity and whose semi-major axis is equal to the radius a, of the cylinder. The neighbouring equilibrium form will be generated by an ellipse of small eccentricity and with the same major axis. Since this ellipse is nearly a circle of radius a, the points where this neighbouring form crosses LM are πa apart in the limit. Let us suppose that the two fixed ends AA, BB of the film are less than πa apart, and that a bulge of the neighbouring equilibrium form is placed symmetrically between AA and BB.

This bulge will evidently pass outside AA and BB. In order to make it pass through them we may shrink it radially towards the centre O of the film in the ratio OB/OC, thus producing a curve which is described by an ellipse of the same eccentricity, but of smaller major and minor axes.

The equilibrium pressure of this last form will be greater than that of the undistorted cylinder, and, therefore, greater than that of the constant pressure supply. The film will, therefore, force gas back into the supply and straighten itself out into the cylindrical form.

A similar argument shows that a small constriction of the same form also straightens itself into the cylindrical form.

If the distance between the ends is greater than πa (AB in the figure), and a bulge of the neighbouring equilibrium form be placed symmetrically between A′ and B′, the curve will pass inside A′ and B′. We may make it pass through these points by expanding it radially, but in so doing we shall have increased the major axis of the generating ellipse, and so reduced the equilibrium pressure. The supply will, therefore, force gas into the bulge, and the latter will increase. Similarly, for a constriction of the same type.

Thus, for this type of displacement the film is stable if its length is less than πa, and unstable if greater.

2. Stability of the cylindrical film which encloses a constant mass of gas.

In considering this problem we require to find the change in volume when the film is displaced. To do this we may first de-

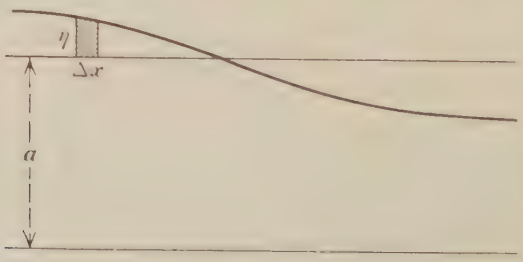


Fig. 4.

termine the change in area of the generating curve when the surface is altered from the cylinder to the neighbouring equilibrium form. The radius of the cylinder being a (Fig. 4), the ordinate at any point of the neighbouring curve may be written $a+\eta$

where η is small. Then the element of the change of area is $\eta\Delta x$, adopting the usual notation. Now, since the loss of curvature due to the change of ordinate from a to $a+\eta$ is compensated by the gain due to curvature of the generating curve, we must have

$$\frac{1}{a}-\frac{1}{a+\eta}=\frac{1}{R},$$

where R is the radius of curvature. This equation gives

$$\frac{\eta}{a(a+\eta)}=\frac{1}{R}, \text{ or } \eta=\frac{a^2}{R} \text{ approximately.}$$

Hence,

$$\eta \,.\, \Delta x=a^2\frac{\Delta x}{R},$$

and $\Delta x/R$ is easily seen to be equal to $\Delta\theta$, neglecting second order quantities, where θ is the inclination of the curve to the axis.

Then, $\eta \,.\, \Delta x=a^2 \,.\, \Delta\theta$, and the total change in area between

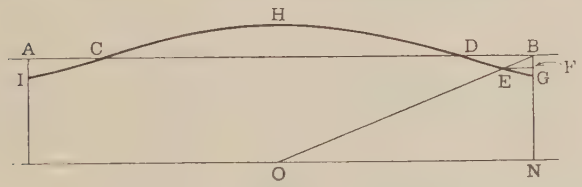


Fig 5.

two points at which the inclinations to the axis are θ_1 and θ_2 is equal to $a^2(\theta_2-\theta_1)$. Since the same value of θ recurs at intervals of $2\pi a$ along the curve, the total change of area will be zero when the length is equal to $2\pi a$.

Again, neglecting quantities of the second order, it follows immediately from this that the volume of the neighbouring equilibrium form of length $2\pi a$ is the same as the volume of the cylinder.

If the film has a length smaller than πa, a symmetrical bulge of the same type as considered before will evidently result in an increase of volume, and, therefore, a decrease of gas pressure, as well as the increase in equilibrium pressure; the tendency for such a bulge to straighten out into the cylindrical form will, therefore, be stronger than in the constant pressure case.

If the length (AB in Fig. 5) is between πa and $2\pi a$, the

volume enclosed by the neighbouring equilibrium form with a bulge symmetrically placed between A and B is greater than that of the cylinder, since the area of the sector CHD is greater than the sum of the portions, each less than half a sector, ACI and DBG.

Expanding the curve radially from O in the ratio $\frac{OB}{OE}\left(=\frac{a'}{a}\right)$, we expand all the ordinates in the same ratio, for $\frac{OB}{OE}=\frac{NB}{NF}=\frac{NB}{NG}$, neglecting the second order quantity FG.

The volume of the film thus displaced will be more than $\left(\frac{a'}{a}\right)^2$ times the volume of the cylinder, and thus the gas pressure will be less than $\left(\frac{a}{a'}\right)^2$ times the original pressure. Since the equilibrium pressure is only decreased in the ratio a/a' times the original, the change in gas pressure more than compensates it.

Fig 6.

A similar argument applies to a constriction of similar form, and thus the film will be stable for this type of displacement up to and evidently beyond a length $2\pi a$. At the length $2\pi a$ (AB in Fig. 6) an unsymmetrical displacement into a neighbouring equilibrium form becomes possible, as shown in the figure, with A and B still on the new curve. The volume enclosed by this displaced film being evidently unchanged, no additional protection is given by Boyle's law, and since both gas pressure and equilibrium pressure are unchanged, the film is evidently in neutral equilibrium as regards small quantities of the first order. If the length (CD in Fig. 6) is greater than $2\pi a$ we may imagine a new displaced form by expanding the neighbouring equilibrium form in the ratio $\frac{CD}{AB}\left(=\lambda, \text{ say}\right)$. This decreases the value of dy/dx at points having the same value of y in the ratio $1/\lambda$, and the value of d^2y/dx^2 in the ratio $1/\lambda^2$. Neglecting second order quantities, therefore, the curvature in the plane of the diagram is reduced in this ratio.

In the neighbouring equilibrium form the curvature in the plane of the diagram just compensates for the change in the value of y from the cylindrical form and therefore in the new displaced form this compensation will be incomplete. The displacement at each point will therefore increase.

If the length of the film is less than $2\pi a$, so that a longitudinal contraction is necessary, the curvature in the plane of the diagram will evidently be increased by the contraction, and will more than compensate for the change in the value of y. The displaced film will, therefore, straighten out into the cylindrical form.

Thus, for this type of displacement the critical length is equal to $2\pi a$—*i.e.*, the unsymmetrical type is a more severe test of stability than the symmetrical.

Since this unsymmetrical displacement involves no change in volume, the argument would apply equally whether the film is closed or connected to a source of gas at constant pressure. For the constant pressure case, therefore, the symmetrical displacement proves to be the more severe test of stability.

The Stability of a Film in the Form of a Catenoid with Equal Ends.

We have seen that the equation to this equilibrium form is $\frac{y}{c}=\cosh\frac{x}{c}$, when the axis of the film is the axis of x and the origin is at the centre of the film. The minimum value of y is evidently equal to c, and this is, therefore, the radius of the "neck" of the film. The tangents from the origin to the catenary are given by the condition $\frac{dy}{dx}=\frac{y}{x}$ at the points of contact with the tangents.

$$\therefore \operatorname{Sinh}\frac{x}{c}=\frac{c}{x}\cosh\frac{x}{c},$$

i.e.,

$$\tanh\frac{x}{c}=\frac{c}{x}.$$

This gives $\frac{x}{c}=\pm 1{\cdot}1996787$, and therefore we get $\frac{y}{x}=\pm 1{\cdot}507879$, and $\frac{y}{c}=\pm 1{\cdot}808971$.

From the form of the equation we see that the whole family of curves for different values of c can be derived by simple radial expansion or reduction of one of them. The tangents from the origin are, therefore, envelopes of the curves, and, of course, two neighbouring curves will cross at their points of contact with the envelope.

If A and B (Fig. 7) represent the trace of the fixed ends of the film we see that two of the catenaries can pass through them; one of these touches the envelope below A and B and the other above.

Let us consider the stability of the latter by assuming the following type of displacement. First, construct a neighbouring catenary of the same type but with slightly smaller

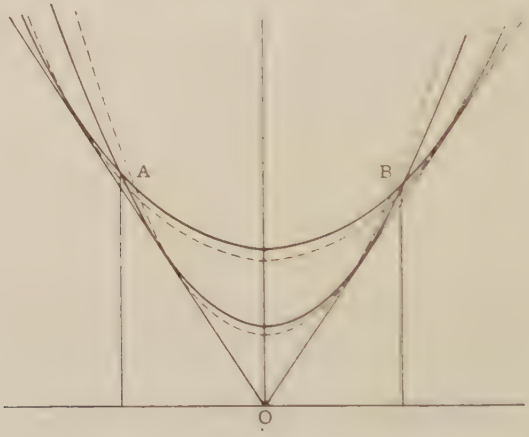


Fig. 7.

parameter c. Since this crosses the original catenary at the points of contact with the envelopes it will pass underneath A and B. Therefore, displace the whole curve upwards until it passes through A and B.

Each point on the new curve will have been displaced downwards between A and B, for the difference between the ordinates of two neighbouring catenaries having the same directrix increases as we pass from their common points to the neck.

After the first step in the displacement the total curvature is still zero, but by the second step the curvature in the plane perpendicular to the plane of the paper is everywhere decreased.

whereas the curvature in the plane of the paper is unaltered. Since the decreased curvature is the inward curvature, the total curvature of the displaced film is everywhere outwards, and, therefore, at every point it moves towards its equilibrium position. For this type of displacement the film is, therefore, stable.

If we consider the same type of displacement for the other catenary the first step in the displacement produces a curve which passes above A and B, and, therefore, the second step is to displace the whole curve downwards. This increases the curvature in the plane normal to the plane of the paper (*i.e.*, the inward curvature), leaving, as before, the other curvature unaltered. The total curvature will therefore be everywhere inwards, and every point of the film will move away from the equilibrium position. This film is therefore unstable. The limiting condition of stability for this type of displacement is evidently that A and B should lie on the envelope, for then the curve still passes through A and B after the first step in the displacement, and, therefore, no second displacement is needed in either direction. The minimum possible value for the radius of the end divided by the half length of the film is therefore given by y/x for the envelope—*i.e.*, 1·507879. In this limiting case the radius of the end divided by the radius of the neck is equal to 1·808971.

Other types of displacement have been investigated, but for none of them does the film become unstable below this limit. Experimentally, we have measured the limiting ratio of the diameter of the end to the length, and have found the value 1·510. This is the same as the value calculated above within the limits of our experimental error.

This Paper makes no claim to prove that the films are stable for all kinds of displacement within the limits calculated. Such a proof requires a much greater knowledge of mathematics in the student. The proof of instability outside certain limits is, however, comparatively simple, for if a film is unstable for a single type of displacement it is unstable. The converse, however, is quite untrue, as is illustrated by some of the displacements investigated, particularly the symmetrical displacement in the case of the cylindrical film.

Enough has been said perhaps to correct the very loose presentation of the subject which has been current up to the present time, especially in English text-books.

ABSTRACT.

The authors give a simple method of calculating the equilibrium form of a thin film which is a surface of revolution.

They then consider the stability for certain kinds of displacement of three classes of such films, viz., the sphere, the cylinder and the catenoid. The mathematics used is quite elementary throughout and the treatment is rigorous.

DISCUSSION.

Prof. J. W. Nicholson thought the Paper was of considerable interest not only to teachers, as suggested by the author, but to physicists in general. He admired the geometrical construction given and the simple way in which the stability or otherwise of a particular form was determined.

XXVI. *On the Theories of Voigt and of Everett Regarding the Origin of Combination Tones.* *By* W. B. Morton, *M.A.*, *and* Mary Darragh, *M.Sc.*, *Queen's University, Belfast.*

Received February 15, 1915.

There has been much discussion on the subject of the additional tones heard when two simple tones are sounded together. This has been concerned with the phenomena to be explained as well as with the theory; the "difference and summation tones" of Helmholtz having been set against the "beat-tones" of Koenig. If mn are the frequencies of the upper and lower primary tones, Helmholtz affirmed the existence of the notes $(m-n)$ $(m+n)$, whereas Koenig heard $(m-hn)$ and $(h+1)\,n-m$, where hn and $(h+1)n$ are the two multiples of n lying on opposite sides of m. Within the compass of an octave Koenig's "lower beat tone" coincides with the difference tone, so that the main subject of dispute concerned the existence of $(m+n)$ on the one hand, and $(2n-m)$ on the other. As a matter of pure observation the question appears to have been settled finally by the recent work of Stumpf * and his school, carried out with unique experimental resources and with the skill derived from long experience in this kind of investigation. The result is to confirm the observations of earlier experimenters, and to establish the existence of both Helmholtz's and Koenig's tones. In addition to these a series of fainter combination-tones can be heard under favourable circumstances when the interval between the primaries is not too large. Inside the interval of a minor third the notes $(2m-n)$ $(3n-2m)$ $(3m-2n)$ $(4n-3m)$ $(4m-3n)$ are heard; between a minor third and a major sixth only the first of these. Between the sixth and the octave only the tones $(m-n)$ $(m+n)$ $(2n-m)$ remain. When the interval exceeds an octave there are only $(m-n)$ $(m+n)$, beyond three octaves only $(m+n)$, and beyond the interval 1 : 12 there are no combination tones at all.

As regards the theory of the phenomena, there has been, and still is, much difference of opinion, and the question is much complicated by the uncertainty as to how far the explanation is to be sought in physical, as distinguished from physiolo-

* "Beiträge zur Akustik u. Musikwissenschaft," Heft V, 1910.

gical or psychological, grounds. The fundamental point at issue between the original theories of Helmholtz and Koenig is the validity of Ohm's law, according to which the ear perceives a note of definite pitch only when a simple harmonic component of corresponding period occurs in the vibratory motion which excites the auditory nerves. When two simple tones are sounded together with amplitudes so small that the principle of superposition holds good, then there are, of course, only these two harmonic components in the air-motion when this is analysed in the Fourier manner, but Helmholtz showed that other components may occur in the motion of the drum of the ear owing to its unsymmetrical structure. His theory relies on the distortion of the aerial motion in the transmitting mechanism of the ear.

Koenig, on the other hand, took a wider view of the capabilities of the ear in recognising periodicity in the incident motion. He supposed that periodic variations of intensity, or "beats," could give the sensation of a tone when sufficiently rapid.

The two theories* mentioned in the title of this Paper may be regarded as extensions of the theories of Koenig and Helmholtz respectively. Voigt goes still further than Koenig in his view regarding the periodicities which the ear can interpret as a tone; Everett seeks to extend the possibilities arising from distortion by the transmitting mechanism.

Both theories receive a good deal of prominence in Auerbach's important work on acoustics (Vol. II. of Winkelmann's "Handbuch der Physik"). An account of Everett's theory is given in Barton's "Text-book of Sound." Some criticisms of Voigt's theory will be found in a Paper by Waetzman,† to which further reference will be made.

The purpose of the present Paper is to point out some objections to Voigt's theory and to give an account of some work which we have done to test the application of the theory of Everett.

Voigt's Theory.

Voigt's theory may be described as a geometrical one. The combination tones are supposed to be determined by certain sine-curves which can be drawn through the extreme points of

* Voigt, "Wied. Ann.," XL., 652, 1890. Everett, "Phil. Mag.," XLI., 199, 1896.

† Waetzmann, "Phys. Zeitschr.," XII., 231, 1911.

the compound vibration-curve representing the motion of the air. It is as if the ear had the faculty of constructing a sine-curve through a number of given points and of recognising the periodicity of the diagram so constructed. Proceeding on this basis, it is shown that curves of the frequencies corresponding to the difference and summation tones can be obtained when the two primary vibrations have the same energy, while the tone $(2n-m)$ is got when the higher primary tone is much weaker than the lower. The special attractiveness of the theory lies in its power to include the Koenig as well as the Helmholtz tones. On this account Auerbach regards it as having finally settled the question, leaving only points of detail for further discussion. His opinion must carry great weight, and it may therefore be worth while to call attention to some difficulties which appear on a close examination of the theory, and which seem to render it untenable as regards the summation tone in particular.

Take, first, the case of equal energies of the primary tones. Their amplitudes are then inversely as their frequencies, and the compound vibration curve has the form

$$y=n \sin mx+m \sin n(x+\delta).$$

It is convenient to take 2π as unit of time and so to call mn the frequencies of the notes.

The stationary points of the curve are given by

$$x=\{(2k+1)\pi+n\delta\}/(m-n),$$

and

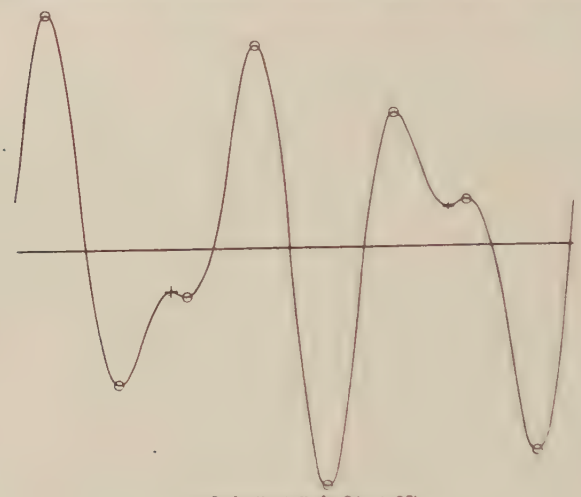
$$x=\{(2k+1)\pi-n\delta\}/(m+n),$$

where k is any integer.

The two sets of points are thus spaced evenly with frequencies $(m-n)$ and $(m+n)$ respectively. We shall call them the $(m-n)$ set and $(m+n)$ set. Each set is now divided into two half-sets by taking alternate points. Through each half of the $(m-n)$ set a sine curve of frequency $\frac{1}{2}(m+n)$ can be drawn, the curves for the two half-sets being opposite in phase. Now, a diagram composed of two equal opposite sine-curves obviously repeats itself in a period half that of the single curves. Therefore, by this construction the points of the $(m-n)$ set are arranged on a diagram which recurs with frequency $(m+n)$. According to Voigt, this fact is perceived by the ear as the summation-tone. In the same way the difference-tone is explained by the similar construction of two sine curves of frequency $\frac{1}{2}(m-n)$ through the two halves of the $(m+n)$ set of

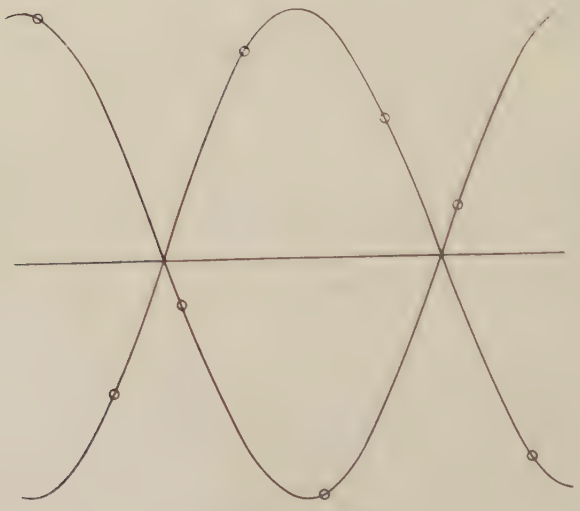
stationary points. The much greater difficulty of hearing the summation-tone is explained by the fact that its curves have to be drawn through a smaller number of points.

Fig. 1 shows the vibration curve for $m=5$, $n=3$, $\delta=6$ deg.



$y=3\sin 5x+5\sin 3(x+6°)$.

FIG. 1.



$y=\pm 5\cos(x-9°)$.

FIG. 2.

The stationary points of the eight-set are marked with circles and those of the two-set with crosses. In Fig. 2 the former set is shown on the sine curves which determine the difference tone 2, and in Fig. 3 the similar construction for the summation tone 8.

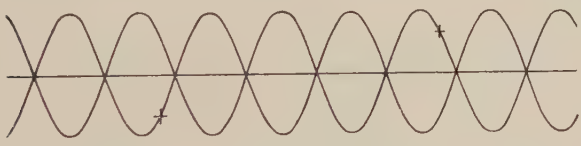
The first objection to be argued against this theory is that a whole series of simple harmonic curves can be drawn through the assigned points in addition to those mentioned above. It is easy to verify that all the points of the $(m+n)$ set lie on all the curves of the family

$$y=(-1)^h(m+n)\sin\{hmx+(h+1)n(x+\delta)\},$$

where h is any integer, positive or negative, while the separate half-sets lie on the pairs of curves given by

$$y=\pm(m+n)\cos\{\tfrac{1}{2}(2h-1)mx+\tfrac{1}{2}(2h+1)n(x+\delta)\}.$$

Taking all the points together, we get in this way the range of



$x=\pm 2\cos(4x+9°)$.

FIG. 3.

frequencies given by the arithmetic progression which has $-m, n$ for two consecutive terms (no regard being paid to sign)—*i.e.*,

$$\ldots -(3m+2n),\quad -(2m+n),\quad -m,\quad n,\quad (m+2n),\quad (2m+3n).$$

On the other hand, if we divide the points into half sets and double the frequencies, as Voigt does, we get the progression with $(m-n)$ for central term and common difference $2(m+n)$—viz., $\ldots -(3m+5n), -(m+3n), (m-n), (3m+n), (5m+3n)\ldots$

Waetzmann (*loc. cit.*) has noticed that the complete group of stationary points lies on curves of the fundamental frequencies m, n and amplitude $(m+n)$, and urges as an objection to Voigt's theory that this should introduce the *octaves* of the primary tones. This does not appear to be valid; it is only when the half-sets are grouped on sine-curves with opposite phases that the frequency should be doubled.

It may be urged in defence of the theory that the difference-tone is singled out by the ear because it corresponds to the

sine-curve of smallest frequency, the primary frequencies m n being left out of account. Even this consideration fails in the case of the summation-tone. The points of the $(m-n)$ set are found to lie on

$$y=(-1)^h(m-n)\sin\{hmx-(h-1)n(x+\delta)\},$$

and the half-sets on

$$y=\pm(m-n)\cos\{\tfrac{1}{2}(2h-1)mx-\tfrac{1}{2}(2h+1)n(x+\delta)\},$$

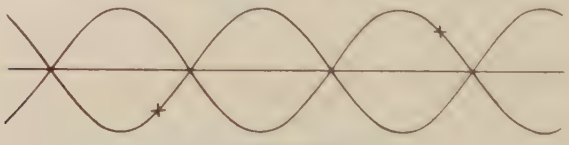
giving the series of frequencies

$$\ldots(3n-2m),\ (2n-m),\ n,\ m,\ (2m-n),\ (3m-2n)\ldots.,$$

and

$$\ldots(5n-3m),\ (3n-m),\ (m+n),\ (3m-n),\ (5m-3n)\ldots.$$

The summation-tone does not, in general, correspond to the lowest frequency in its series. For example, in the case of the interval $3:5$ we have $3n-m=4$. It is difficult to see any reason why the ear should construct the curve of frequency 8 through the two determining points as shown in Fig. 3 rather than the curve of frequency 4, as in Fig. 4, if, indeed, it is con-



$y=+2\cos(2x+27°)$.

Fig. 4.

ceivable that any curve having a physical significance should be determined by such scanty data.

Another objection to this way of explaining the summation-tone lies in the fact that the $(m-n)$ stationary points supposed to determine it are minima of the elongation from the mean position, whereas the $(m+n)$ points for the difference tone are maxima. We have

$$\frac{d^2y}{dx^2}=-mn\{m\sin mx+n\sin n(x+\delta)\}.$$

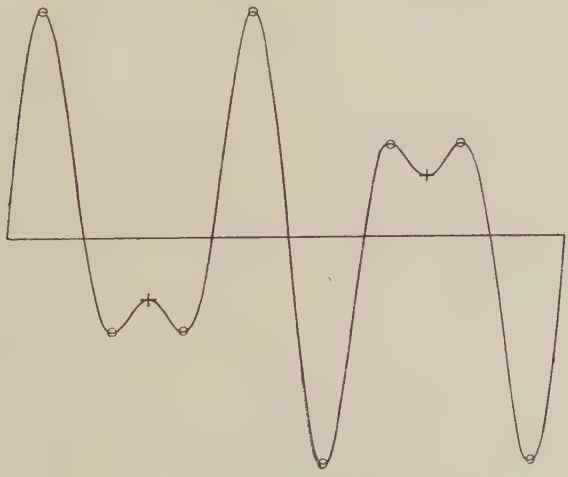
At a point of the $(m-n)$ set $\sin mx=-\sin n(x+\delta)$, which gives $\dfrac{d^2y}{dx^2}=mny$.

So when y is positive its value is a minimum, and when it is

negative the value is a maximum; in either case we have a point of nearest approach to the axis. For points of the $(m+n)$ set, on the other hand, $\frac{d^2y}{dx^2}=-mny$. It seems scarcely likely that the two classes of points should both give rise to tone-sensations, differing only in intensity.

Again, it appears to be a necessary requirement of a satisfactory theory that it should hold equally well for different phase-relationships of the two primary vibrations. It should also hold through a fairly wide range of relative intensities of the primary sounds. Voigt's theory does not stand these tests.

The effect of varying the phase angle, keeping the energies equal, is shown on Figs. 5, 1 and 6, corresponding to $\delta=0, 6°, 12°$,



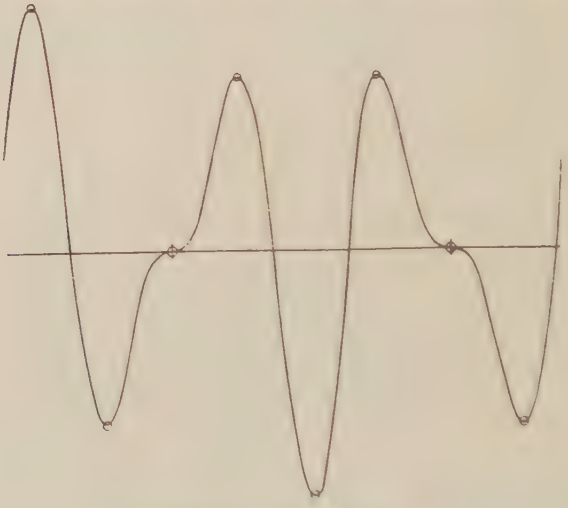
$y=3\sin 5x+5\sin 3x.$

FIG. 5.

for the interval 3 : 5. All the different shapes of the compound curve are obtained as δ increases from 0 to π/mn. It will be seen that the points of the $(m-n)$ set marked as before with a cross, begin by being midway between a pair of points of the other set. In this case their ordinates are most different from those of the adjacent stationary points. With increasing δ the $(m-n)$ points move up to neighbouring points of the other set, fusing with them on the axis for $\delta=\pi/mn$. In this last case, of course, the points are not available for determining any

curve, and in general it is clear that the distinctness of the summation-tone would vary greatly with changes of δ.

When the energies of the primary vibrations are unequal the positions of the stationary points on the curve cannot be expressed by a simple formula, but the progressive modification of the form of the curve can be followed by tracing what becomes of the $(m-n)$ minima, starting from the equal-energy case. If the energy of the upper tone is increased the indentation at each minimum becomes deeper until the axis is reached and crossed, so that the point becomes a maximum on the opposite side. Meanwhile the former maxima persist, so that in the end we get $(m+n)+(m-n)=2m$ peaks on the curve



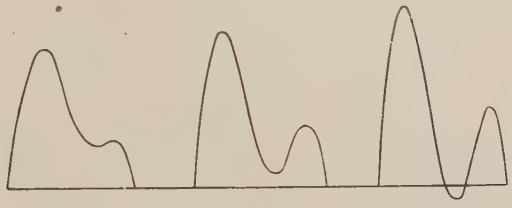
$y=3\sin 5x+5\sin 3(x+12^{\circ})$.

FIG. 6.

corresponding to the curve for the upper note alone. This change is shown on Fig. 7. On the other hand, if the amplitude of the lower note is increased, the depressions at the minima are smoothed out, carrying along with them in each case an adjacent maximum. The combination of minimum and maximum is replaced by a kind of shoulder on the curve which becomes less and less conspicuous, as the upper note is made relatively weaker (Fig. 8). Ultimately the curve has $(m+n)-(m-n)=2n$ peaks, the number for the lower tone. It will be seen that the stationary points supposed to deter-

mine the summation-tone cease to exist when the ratio of the energies is but little different from unity.

The $(m-n)$ minima, and the shoulders which replace them, divide the peaks of the curve into groups, which thus recur with the frequency of the difference-tone. In this grouping we have evidently what remains of the " beats " shown on the



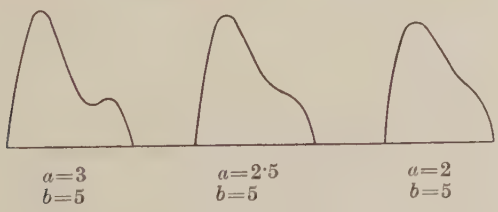
$a=3$ $b=5$ | $a=4$ $b=5$ | $a=5$ $b=5$

$y=a \sin 5x+b \sin 3(x+6°)$.

FIG. 7.

compound curve for two notes of nearly the same pitch, so that we have really come back to Koenig's original explanation of the difference-tones.

It may be mentioned that there is no obvious grouping of the peaks to correspond to Koenig's tone $(2n-m)$, except in cases where $(2n-m)=1$, so that the complete curve is repeated in this period. For example, when the curve for the ratio 5 : 7 is



$a=3$ $b=5$ | $a=2{\cdot}5$ $b=5$ | $a=2$ $b=5$

$y=a \sin 5x+b \sin 3(x+6°)$.

FIG. 8.

drawn the peaks fall into two groups in each complete period, separated by minima, but there is no grouping into three.

Passing now to the other special case considered by Voigt when the upper tone is very weak compared with the lower, it is assumed that the peaks of the compound vibration-curve have the same *positions* as the maxima of the curve of the

lower note alone, the presence of the weak upper tone merely altering the *heights* of the maxima. Thus, the peaks vary above and below their mean level. The deviations from this level are given by the magnitudes of the ordinates of the sine-curve for the upper note, taken at the positions of the maxima of the lower note. In other words, we take a series of equidistant points, spaced with frequency n, on a sine-curve of frequency m. Voigt shows that a sine-curve of frequency $(2n-m)$ can be run through these points, and to this geometrical fact he ascribes the presence of Koenig's second beat-tone.

In the first place it seems doubtful whether one can ignore the displacement of the maxima while taking into account their change of height. If we represent the compound curve by

$$y=c \sin mx+\sin n(x+\delta),$$

where c is small compared with unity, we find that the positions of the maxima are given, to the first order of approximation by

$$n(x+\delta)=(2h+1)\frac{\pi}{2}-\frac{m}{n}c \cos \frac{m}{n}\left\{(2h+1)\frac{\pi}{2}-n\delta\right\},$$

and the maximum ordinate by

$$y=1+c \sin \frac{m}{n}\left\{(2h+1)\frac{\pi}{2}-n\delta\right\},$$

so that the two effects in question are in general of the same order of magnitude.

But if this point be waived we have still the difficulty that other curves can be drawn through the determining points which do not correspond to tones actually heard. In fact, all the curves of the family $y=\sin\{(kn+m)x+\delta\}$ pass through the points where the curve $y=\sin(mx+\delta)$ is met by the ordinates at $x=0, \frac{2\pi}{n}, \frac{4\pi}{n}$, &c., k being any positive or negative integer. Thus, for the interval 3 : 5 we should have the series of frequencies 7, 4, 1, 2, 8, 11.

In view of these difficulties, it seems impossible to regard Voigt's theory as satisfactory, except possibly for the difference-tone, where it may be regarded as a definite form of Koenig's original theory. Further, the trend of recent work is in favour of the validity of Ohm's law. No clear case has been

established in which a tone is given by a mere periodicity, apart from the presence of a simple harmonic component in the analysis of the vibrations.

On the other hand, it should be pointed out that Voigt's theory has since the date of its publication received some experimental support from the fact discovered by Meyer, and confirmed with some modification by Stumpf—namely, that the difference-tone is more easily heard when the lower note is weak and the beat-tone when it is strong. Another supposed point in its favour has been mentioned by Waetzmann, who finds in it an explanation of the fact that difference-tones are not heard for intervals above the octave, when they lie between the pitches of the primary tones. Waetzmann draws the vibration-curve for 1 : 3, and points out that the $(m+n)=4$ maxima, which determine the difference-tone are all of the same height, and would not, therefore, be likely to impress the ear and give rise to a tone. However, this conclusion depends on the fact that the curve is drawn with no phase-difference between the primary vibrations. If the more general case $y=3\sin x+\sin 3(x+\delta)$ is plotted for different values of δ, it is found that the maxima are unequal and that the division of the maxima into groups by the minima or "shoulders" on the curve still occurs.

Everett's Theory.

In Everett's theory stress is laid on the distortion which the vibration-curve of the air-motion must undergo in its passage through the mechanism of the ear. The analysing function of the ear is applied to the vibrations at the inner end of this mechanism, so the result of the analysis will be affected by the distortion. In Everett's opinion the frequency most likely to be introduced is the highest common measure of the primary frequencies; the tone of low pitch of which both primaries are harmonics, having the period in which the complete vibration-curve is repeated. When one analyses a periodic curve of any random form one generally finds the fundamental component coming out strongly. Curves built up of two simple harmonic curves are exceptions to this rule, but if the curve is knocked out of shape in any manner and then analysed the exceptional property of not containing the fundamental would be lost, and this component would generally appear in the analysis.

The experimental work has not confirmed this conclusion of

Everett's. The common fundamental is only heard when it happens to coincide with a difference or beat-tone. Notwithstanding this, the suggestion of Everett seems worthy of further investigation. It is beyond doubt that some distortion takes place, and has an important effect on the quality of the tone heard.

One way of testing the matter is to assume a certain definite type of distortion, and to examine by actual analysis the new frequencies which it introduces into the vibration-curve. In seeking for a reasonable hypothesis to work on one naturally thinks of giving weight to the asymmetry of the transmitting mechanism, in virtue of which it must treat differently the displacements in opposite directions. A simple assumption, and one which lends itself readily to examination in detail, is that displacements in one direction are reduced in a definite ratio as compared with those in the opposite direction. It is not suggested that any definite mechanical arrangement will produce exactly this alteration in the incident vibrations. All the evidence goes to show that the actual procedure in audition is very complicated, depending on physical, physiological and psychical factors. There may, therefore, be some justification for the purely geometrical method here proposed which keeps clear of any special assumption of a mechanical kind. It is perhaps allowable to say that something like this kind of distortion will be caused by a transmission process which gives differential treatment to the inward and outward movements of the ear-drum.

The proposed modification of the vibration-curve may be effected by paring off a certain small fraction from, say, all the positive ordinates. This is the same thing as adding to the given curve another of small amplitude, which is similar to the given curve, with reversed sign, when the latter lies above the axis, and which runs along the axis when the given curve passes below. The new frequencies introduced by the distortion will be obtained by the analysis of this added curve. The actual amplitudes of the new vibrations will, of course, depend on the amount of distortion—*i.e.*, on the ratio in which the positive ordinates are reduced. But their relative amplitudes can obviously be got by applying harmonic analysis to the positive parts of the original vibration-curve, replacing its negative parts by pieces of the axis. We shall arrive at the same result if the negative parts of the curve are taken instead of the positive.

If $x=a_1$, a_2, a_3, &c., are the zeroes of

$$y=a \sin mx+b \sin n(x+\delta),$$

then we require the Fourier expansion of $f(x)$, defined by $f(x)=y$, when x lies in one of the intervals a_1 to a_2, a_3 to a_4, &c., and $f(x)=0$, when x lies in one of the intervals a_2 to a_3, a_4 to a_5, &c.

This process applied to a single vibration gives the manner in which overtones would be produced by the supposed distortion. The expansion of the function

$$f(x)=\sin x. \qquad 0 \angle x \angle \pi.$$
$$f(x)=0. \qquad \pi \angle x \angle 2\pi$$

is

$$\frac{1}{\pi}+\tfrac{1}{2}\sin x-\frac{2}{3\pi}\cos 2x-\frac{2}{15\pi}m4x \ldots -\frac{2}{(4r^2-1)\pi}\cos 2rx-,$$

showing the possibility of the introduction of the octave into the auditory effect produced by physically pure tones.

In the case of a compound curve the calculation of the coefficients becomes rather laborious, especially for the larger frequency numbers. A mechanical analyser (Stärzl type) was used in the first instance, and the more important cases were afterwards checked by calculation. The primaries were first taken with equal energies, and the following results were found to hold for all values of the phase-difference δ. The components of the two primary frequencies appear with about half their original amplitudes. The difference and summation tones are both prominent. Next to them comes the octave of the lower primary tone. Neither the second beat-tone nor the common fundamental is specially prominent.

The following table gives the amplitudes of the successive components of the analysis applied to $y=n \sin mx+m \sin nx$— *i.e.*, with $\delta=0$, and equal energies :—

$m:n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1 : 1	1·00	0·42	0	0·03	0	0·06	0	0·02	0	...	...	...	...	...
2 : 1	1·09	0·66	0·26	0·09	0·06	0·04	0·03	0·02	0·02	...	...	...	...	...
3 : 2	0·65	1·51	1·13	0·45	0·57	0·14	0	0	0·12	...	...	...	...	...
4 : 3	0·92	0·19	2·00	1·50	0·17	0·56	0·86	0·23	0·04	...	...	...	...	...
5 : 2	0·21	2·50	0·65	0·96	1·00	0·10	0·60	0·13	0	...	...	...	...	...
5 : 3	0	0·82	2·50	0·06	1·50	0·74	0	0·95	0	...	...	...	...	...
5 : 4	1·19	0·23	0·14	2·46	2·01	0·11	0·14	0·56	1·17	...	...	...	...	...
6 : 5	1·51	0·20	0·11	0·06	2·89	2·58	0·09	0·12	0·22	...	1·20	...	...	...
7 : 4	0·17	0·08	1·36	3·50	0·36	0·09	2·05	1·07	0	...	1·09	...	...	...
7 : 5	0	1·44	0	0·28	3·50	0	2·50	0·49	0	0 94	0	1·50	...	...
8 : 5	0·07	0·07	1·53	0·05	3·99	0·21	0·33	2·51	0·19	...	...	...	1·37	...
9 : 5	0·02	0·06	0	1·57	4·50	0·36	0	0·14	2·50	...	...	...	...	2·92

Introduction of a phase-difference made only a slight difference in the relative magnitudes of the components. For example, for the ratio 5 : 3 the successive terms in the analysis of $y=3\sin 5x+5\cos 3x$ are

$$0,\ 0{\cdot}87,\ 2{\cdot}50,\ 0{\cdot}21,\ 1{\cdot}50,\ 0{\cdot}78,\ 0,\ 0{\cdot}91,\ 0,$$

which do not differ much from the numbers given in the table for $3\sin 5x+5\sin 3x$.

It is not difficult to see the mathematical reason for the comparative largeness of the terms of frequencies $m-n$, $(m+n)$ and $2n$. The expressions for the terms of frequency r in the analysis of the positive parts of the curve

$$y=a\sin mx+b\sin n(x+\delta)\ \text{are}$$

$$\sin rx\left\{\frac{a}{2\pi}\left[\frac{\sin(m-r)x}{m-r}-\frac{\sin(m+r)x}{m+r}\right]\right.$$
$$\left.+\frac{b}{2\pi}\left[\frac{\sin(n-r)x+n\delta}{n-r}-\frac{\sin(n+r)x+n\delta}{n+r}\right]\right\}$$
$$-\cos rx\left\{\frac{a}{2\pi}\left[\frac{\cos(m-r)x}{m-r}+\frac{\cos(m+r)x}{m+r}\right]\right.$$
$$\left.+\frac{b}{2\pi}\left[\frac{\cos(n-r)x+n\delta}{n-r}+\frac{\cos(n+r)x-n\delta}{n+r}\right]\right\},$$

where the expressions in square brackets are taken between the limits corresponding to the positive values of y. Thus, the quantities entering are of the form $(-\sin pa_1+\sin pa_2-\ldots)\,p$ and the same thing with cosines, where y is positive between $x=a_1$ and $x=a_2$, and so on. When these expressions are evaluated it is found that the greatest numerical value is got for either the sine or the cosine expression when $p=n$.

To see the reason for this it is to be noticed that in the cases considered the number of the roots corresponds to the case of the lower note by itself, the positions of the roots being displaced in consequence of the presence of the upper note. If, now, the lower note were alone we should have $a_2=a_1+\frac{\pi}{n}$, $a_3=a_1+\frac{2\pi}{n}$, &c., so $na_2=na_1+\pi$, $na_3=na_1+2\pi$, &c. Accordingly all the terms would have the same sign and magnitude, and the expressions would reduce to $-2n\sin na$ and $-2n\cos na$ respectively. If the phase-angle δ is such that one of these is small, the other will be large. If the integer p is different from n

some of the terms in the expressions will be positive and others negative, so that the sum is comparatively small. Of course, coincidence of sign is got again when $p=3n$, but the value is reduced by the division by p. When the note m is added the roots are displaced, but when the energies are equal the amplitude of the upper note is less, and so the positions of the roots continue to be fairly evenly spaced along the range 0 to 2π, and the above argument still applies. Therefore, we should expect to find comparatively large Fourier coefficients in cases where one of the numbers $(m-r)$ $(n-r)$ has the value $\pm n$, *i.e.*, when $r=m+n$, $m-n$ or $2n$.

The prominence of these frequencies persists when the energies of the primaries are made unequal. To obtain a measure of the relative loudness of, say, the difference-tone we may divide the square of its amplitude by $(a^2m^2+b^2n^2)$,

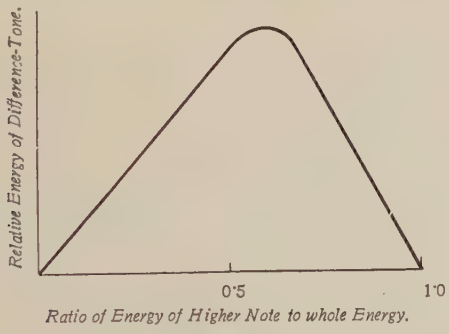


FIG. 9

representing the total energy of the two primaries together. Estimated in this way the maximum value is found when the higher primary tone is somewhat louder than the other. The curve (Fig. 9) shows the variation of the relative loudness of the difference-tone for the interval 3 : 5, the abscissa being the ratio of the energy of the upper primary to the total energy of both primaries. The curve for the summation tone follows a similar course.

It will be seen that the result of this particular kind of distortion affords no support for Everett's contention that the common fundamental would be the most prominent tone introduced. In this it is in agreement with the facts of observation.

On the other hand, no help is given to the theory on the

points which were sources of difficulty in the original explanation of Helmholtz—viz :—

1. The comparative weakness of the summation-tone. Here, as in Helmholtz's theory, the difference and summation-tones appear with amplitudes of the same order of magnitude, so the latter, having greater frequency, should have greater physical energy.

2. The disappearance of the difference-tone with wide intervals. Its relative energy, measured as explained above, rather increases as the interval ratio becomes greater.

3. The presence of the second beat-tone of Koenig. The frequency $(2n-m)$ does not appear in the analysis of the compound curve when the upper note is made relatively weak. There is, of course, the possibility that the explanations of these three points are to be sought in other than purely physical considerations.

Another kind of distortion which suggests itself is that which might arise from slipping at a contact in the transmitting mechanism; for example, between the "hammer" and "anvil" bones. If, in a movement in one direction, there were a small amount of slip before the second bone followed the motion of the first, the effect would be to decrease, say, all the positive ordinates by a definite amount. The representative point on reaching the axis from below would first slide along it for a short distance, and then follow a course parallel to the undistorted curve. We have applied the method of harmonic analysis to this case also, but have found no results of general application.

ABSTRACT.

Voigt connects the existence of difference and summation tones with the fact that the stationary points of the compound vibration-curve, when the primary tones have equal energies, can be grouped in a certain way on sine curves, which recur in the periods of these combination tones. As against this view it is urged (1) that the same points can equally well be grouped on a whole series of curves with other frequencies; (2) that the distinctness of the combination-tones would on this theory vary greatly with phase-relationship of the primaries; (3) that the tones would disappear when the energies of the primaries are not very unequal. Voigt applies a similar method to the case where the upper tone is weak compared to the lower to account for Koenig's second beat-tone. The first of the above objections again applies.

Everett supposed that the distortion of the vibration-curve in passing through the ear would bring in the note whose frequency is the highest common factor of the primary frequencies. The effect

of a special kind of distortion has been tested, consisting in a proportional reduction of ordinates in one direction. The result does not confirm Everett's contention, but shows the appearance of the summation and difference tones.

DISCUSSION.

Mr. F. J. W. Whipple thought the matter was an extremely interesting one, and that the type of distortion assumed by the authors was very feasible. He suggested that an amplification of the theory might be applicable to musical instruments. For example, much of the beauty of the tones of a violin was attributed to the presence of many additional notes besides the fundamental. Quite possibly an asymmetry in the vibrations of the sounding board was responsible for the addition of combination tones of the higher harmonics.

Mr. D. Owen referred to the case of a man who had no ear-drums, but could nevertheless hear combination tones, which Prof. Morton had cited as an instance in which unsymmetrical vibration of the receiving system could not be the cause of these tones. He presumed that this subject was very deaf, and that any sounds which he could hear at all would be very loud. Consequently, there would be violent displacements of the bones of the ear, and one would expect considerable asymmetry in their vibrations.

Dr. Russell thought that the authors had without doubt completely demolished Voigt's theory, which seemed to have no physical basis whatever. With regard to the unsymmetrical vibrations of the drum, he supposed Helmholtz in his investigation had treated the tympanum as a thin plate, but the attached bones and adjacent fluid would render the problem much more complex.

Prof. S. P. Thompson communicated: Though it is well to discuss the theory of combination tones it is much more important to be certain first *what are the facts*, to account for which some theory is required. Hence I raise the vital question of what the facts are: According to von Helmholtz, the combination tones are of two kinds, differential and summational. That is to say, when two tones of respective frequencies m and n are simultaneously sounded, the ear hears the *difference* tone, of frequency $m-n$, and the *summation* tone, of frequency $m+n$; and these are ear-manufactured, and cannot be heard in any resonator, and have nothing to do with beats. According to Koenig, the combination tones are simply beat-tones (and were so regarded also by Thomas Young), and cannot be heard in any resonator. If any supposed combination tone is heard in a resonator, that is a proof that it exists objectively in one or other of the two primary tones, and that, therefore, one or both of the two primary tones is not a *pure* tone but contains the alleged combination tone as a harmonic. Koenig found that beats and beat-tones fall into two series (called superior and inferior), corresponding respectively to the two remainders—positive and negative—to be found by dividing the frequency of the higher tone by that of the lower tone. Thus, let the two primary frequencies be 40 and 74. Then, if we divide 74 by 40, it goes once with a positive remainder of 34, or it goes twice with a negative remainder of 6. Koenig heard, in these circumstances, a rapid beat (inferior) of 34 and a slow beat (superior) of 6. If we take as primaries the note of frequency 3,328 and the note 2,048, there will be heard, according to von Helmholtz, the difference tone 1,280, and the summation tone 5,376. According to Koenig, there will be heard the (inferior) positive remainder 1,280 and the (superior) negative remainder 768. As a matter of fact, two tones are heard, the stronger being 768, and the weaker 1,280. The summation tone 5,376 is not heard at all. If we take, again, two forks giving

2,304 and 1,024, the only combination-tone heard is 256, which is neither difference nor sum. It is the (positive) remainder heard by dividing 2,304 by 1,024. Voigt's theory accounts for this, von Helmholtz's does not. Koenig, who in his life-time tuned with his own hands some tens of thousands of forks, and used the beat-tones to determine the tuning of his high-frequency forks of inaudible pitch, had never been able to hear any tones corresponding to the summational numbers. They simply do not exist if the primary tones are pure.

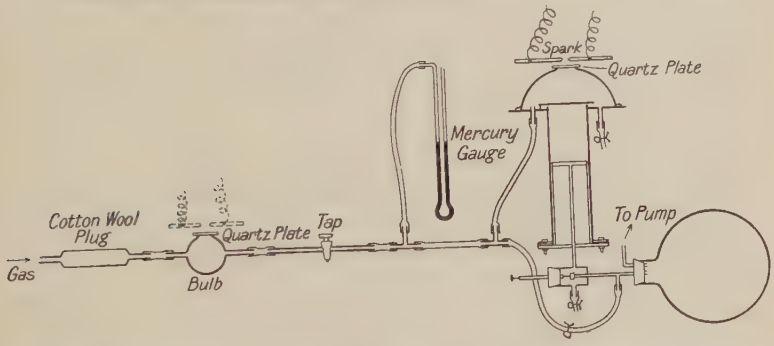
Prof. Morton, in reply, said he could not say how far combination tones might be produced by the sounding board of a violin. They certainly could be produced by the membrane of a telephone receiver. He had not quoted the man with no ear drums as a case in which asymmetry was not responsible for these tones, but only as one in which we had to go further back than the tympanum for the seat of this asymmetry. A German author had tried to locate this in the motion of the fluids of the ear. Helmholtz's investigation of the vibration of the drum was even less satisfactory than the chairman had supposed, as he treated it simply as an oscillating particle. The analysis had more recently been extended to the case of a membrane and similar results obtained, but even this was still remote from the actual case of the ear drum. With reference to Prof. S. P. Thompson's remarks, the work of observers since the time of Koenig had confirmed his statements as to the presence of the beat-tones; but not his denial of the summation-tones.

XXVII. *Experiments on Condensation Nuclei produced in Gases by Ultra-violet Light.* *By* Miss Maud Saltmarsh, *Bedford College.**

Received March 2, 1915.

The experiments described in this Paper were performed in the years 1907 and 1914. Those on the action of an electric field were carried out at the Cavendish Laboratory, Cambridge, in 1907, and the others at Bedford College, Regent's Park, in the summer of 1914. During this interval a very exhaustive research on the action of ultra-violet light on gases was made by Lenard and Ramsauer, and published by them in the "Sitzungberichte" of the Heidelberg University, and it was not until after the completion of these observations that their Paper was brought to my notice.

Lenard and Ramsauer used a steam jet to observe condensa-



tion, and this is a less sensitive means of detecting nuclei than the expansion apparatus used throughout these observations, and therefore it was thought that a short account of them might be of value.

The Wilson expansion apparatus used is depicted in the diagram. Its principle is well known, and has been fully described by Mr. C. T. R. Wilson,† so that a further description of it is unnecessary. The cloud chamber, however, was different in the present case. In the experiments on the electrical nature of the nuclei, it consisted of the bottom portion of a cylindrical bell jar, 19·1 cm. in diameter and 5·1 cm. high. The edges of it were ground flat, and to the top one a brass plate was sealed. The other edge rested on a rubber ring, placed on the base of the cloud chamber, and

* Communicated by W. Eccles.

† C. T. R. Wilson, "Phil. Mag.," June, 1904, p. 686.

formed with it an air-tight junction. There was a hole 1·8 cm. in diameter in the brass plate, and this was covered by a quartz plate 0·5 mm. thick, through which the ultra-violet light could pass. An electric field could be put on between this disc and another inside the cloud chamber fixed parallel to it, and insulated from the rest of the apparatus.

The source of ultra-violet light was a spark about 3 mm. long between zinc terminals. An induction coil giving a 6 in. spark was used with three or four accumulators, and a Leyden jar was put in the secondary.

The expansion method of observing the condensation is very sensitive; the presence of only one or two nuclei which are "caught" by any given expansion can be easily seen. A fog contains many more drops (and hence nuclei) than a shower, and the drops hang for some time in the air, while the shower falls quickly. In the electrical observations there was an air space of 6 or more centimetres between the spark and the quartz window, and therefore the ultra-violet rays, which are strongly absorbed by air, were cut out.

§ 1. To observe the effect of an electric field a comparison was made between the behaviour of the nuclei in the field and that of ions produced by radium; and, in order to facilitate the motion of the nuclei or ions, the apparatus was filled with hydrogen which contained a small proportion of air.

Even without an electric field, the ions were less effective in producing condensation than the nuclei, and a shower was formed on the latter, when the expansion was too small to "catch" any of the ions. The least expansion required for condensation on negative ions is about 155 mm. of mercury.

There was no marked increase in the density of the cloud on the nuclei, as the expansions were successively increased through this value, which might have been the case had there been many ions present.

The differing effects of the field, which was 50 volts per centimetre on ions and on the nuclei, was very marked with an expansion of 189·5 mm., as shown by the following table:—

Difference of pressure on expansion.	Nature of cloud.					
	No light or radium.		Radium.		Ultra-violet light.	
	P.D. = 0.	P.D. = 50 volts per cm.	P.D. = 0.	P.D. = 50 volts per cm.	P.D. = 0.	P.D. = 50 volts per cm.
189·5 mm.	Large shower	Very small shower	Fog	Large shower	Very large shower	Very large shower

Observations of a similar kind were made, and in none of them were the nuclei appreciably moved by the field.

That the ultra-violet nuclei do not disappear more quickly in the presence of an electric field than without one was demonstrated as follows :—

The time of exposure to the light was arranged so that a very dense shower was produced on the nuclei with an expansion of 172 mm. This shower was the same in the presence or absence of the field. Making the expansion 2·5 minutes after stopping the spark a shower was obtained, and after four minutes a smaller one, but in both case the number of drops was the same, whether the field was on or whether it was not.

With the same expansion and without an electric field a shower of about the same density was produced on ions formed by exposure to radium of suitable duration. If there was no field, and the apparatus was left for four minutes after the radium had been removed, a shower was obtained on expansion of about the same density as that on the nuclei after four minutes ; but in the presence of a field the number of drops formed on the ions was very small after $\frac{6}{7}$ second.

The rates of disappearance of the nuclei and ions seemed to be about the same when there was no electric field ; but, while it had no effect on the rate of disappearance of the nuclei, it increased that of the ions very much.

Lenard and Ramsauer* have divided the action of ultra-violet light on gases into three separate divisions :—

1. The production of electrified particles, sometimes of molecular dimensions and sometimes larger, the latter being formed by ions of molecular size joining on to uncharged nuclei.

2. Chemical action, such as the formation of ozone.

3. Production of nuclei, which serve as centres of condensation.

They showed by electrometer measurements that ultra-violet light, which is highly absorbed by air, is very strongly photo-electric in air, and the permanent gases of it, also in CO_2, NH_3, CS_2, &c.

Ultra-violet light of longer wave-length which passes through several centimetres of air is not photo-electric in the permanent gases of air if they are pure, but is so in atmospheric air, since this always contains other minor constituents, by which the light is absorbed (CO_2, NH_3, &c.). They also main-

* Lenard and Ramsauer, Heidelberg " Berichte," 1910 and 1911.

tain that the nuclei produced in gases by ultra-violet light are much more effective for condensation than gaseous ions.

Accordingly, if both ions and nuclei be formed by the ultra-violet light, the nuclei would be "caught" rather than the ions, and the density of the cloud might not be much affected by the removal of the ions. This would account for the cloud on the nuclei being as dense in the presence of a field as without one.

If, however, some ions were formed by the light, and they joined on to the nuclei to form the larger carriers observed by Lenard and Ramsauer, they should disappear more rapidly when acted on by an electric field than when they merely diffused away. This was not found to be the case, but possibly not many ions were formed by the light.

§ 2. In the following series of experiments another expansion apparatus was used, but it was essentially the same as before, the only difference being in the cloud chamber. This was a 5 in. dialyser, the large edge of which was ground so as to form with vaseline or rubber lubricant an air tight junction with the brass plate forming the base of the cloud chamber. The small neck of the dialyser was cut off short, and a quartz plate 0·1 cm. thick was sealed to it.

Sometimes the gas experimented upon was collected in a small glass bulb, shown in the diagram. It was about 5 cm. in diameter and had a quartz window, so that the gas could be exposed in it to the action of the light, and could then be drawn into the cloud chamber.

Observations were made with the vapours of other liquids besides water in order to determine if the nuclei were equally effective in producing condensation of them. The water in the expansion apparatus was in each case replaced by the liquid to be experimented on.

The values in the first table give the least expansion required for condensation in dust-free air. This occurs on the negative ions, which seem always to be present in small numbers in atmospheric air.*

Water	$v_2/v_1=1{\cdot}25$
Absolute alcohol	$v_2/v_1=1{\cdot}16$
Toluol	$v_2/v_1=1{\cdot}29$
Turpentine	$v_2/v_1=1{\cdot}30$

* C. T. R. Wilson, "Phil. Mag.," June, 1904.

In observing condensation on the nuclei, in order that they might be exactly the same in all cases and not dependent on the nature of the vapour, dust-free atmospheric air was exposed in the bulb to ultra-violet light from the spark. Since the bulb had not before been in communication with the liquids, it contained none of their vapours, and the nuclei were therefore identical in all cases. They were swept from the bulb into the cloud chamber, in which the air was at diminished pressure, by causing a stream of dust-free air to be drawn through the bulb into the cloud chamber, until the requisite pressure was obtained. The spark was placed near the quartz window of the bulb about 0·5 cm. from it; the light was, therefore, fairly strong and the supply of nuclei plentiful.

From the table it will be seen that the nuclei were effective in producing condensation of all these vapours, and require only small expansions for condensation to occur. Almost the smallest expansion possible to give was sufficient to produce some drops.

Vapour.	Time of exposure.	Difference of pressure on expansion.	Cloud.
Abs. alcohol	15 secs.	2·9 cm.	Shower
Toluol	1 min.	3·1 „	Thin fog
Turpentine	1 min.	5·3 „	Thin fog

Since the nuclei are equally effective in causing condensation of the vapours of water, alcohol, toluol and turpentine, it does not seem likely that the substance of which they consist should, by dissolving in a minute drop of the liquid, lower the vapour pressure, and hence help the drop to grow. The substance would have to be equally soluble in all four liquids. This explanation of the nuclei has been put forward, and a particular case of it is dealt with later on in the Paper.

The effect of exposing air saturated with the various vapours in turn to the direct action of ultra-violet light was then tried. It is well known that if air saturated with water vapour is exposed to the action of very strong ultra-violet light, a dense fog appears in it after a time without any expansion having been made. The ultra-violet light must be very strong to produce this effect. In the present case the effect was not obtained with water vapour even when the spark was placed

as close as possible to the quartz plate, but it was with alcohol vapour.

Spontaneous condensation of alcohol vapour seems to occur on very small nuclei, if one can judge of the size of them by the strength of the light which forms them; but even water vapour would condense with a small expansion on nuclei formed by light which had traversed 50 cm. of atmospheric air between the spark and the quartz plate.

The following table gives some of the observations made with direct exposure of the air in the cloud vessel :—

Vapour.	Time of exposure.	Difference of pressure on exposure.	Distance of spark.	Cloud.
Water	15 secs.	1·7 cm.	0·7 cm.	Thin fog
Water	15 secs.	1·8 cm.	50 cm.	Small fog
Abs. alcohol ...	1 min.	No expansion	0·7 cm.	Fog
Abs. alcohol ...	1 sec.	5·7 cm.	50 cm.	Small shower
Toluol	15 secs.	8·45 „	0·7 „	Large shower
Turpentine ...	1 min.	3·5 „	0·7 „	Thin fog

§ 3. The action of the light to form nuclei in various gases was then observed, condensation of water vapour on them being used to detect the nuclei. In order to determine if the presence of water vapour was necessary for the production of nuclei, dust-free air was passed over H_2SO_4 on pumice stone into the bulb, and left there in contact with P_2O_5 for some time. It was then exposed to the action of the light, and the nuclei were swept into the cloud chambers as before.

Time of contact with P_2O_5	Difference of pressure on expansion	Cloud.
Three hours	8·1 cm.	Very large shower
Two days	8·05 „	Fairly large shower
Air not dried	11·85 „	Very large shower

It is not probable that the very last traces of moisture were removed, but the formation of nuclei did not seem to be dependent on the presence of more than a very minute quantity of water vapour.

Carbon dioxide was then tried, and in this case the cloud vessel was exhausted and filled several times over with CO_2 from a cylinder of liquid CO_2, and the gas was exposed in it to

the light. With an exposure of one minute duration and a difference of pressure of 12·7 cm. on expansion, a dense fog was obtained exactly similar to one obtained in air under the same conditions.

CO_2 and air, probably the oxygen of it, therefore, are equally susceptible to the action of ultra-violet light to produce nuclei.

The next three gases experimented upon showed little or no nuclei formation. First of all, the bulb was filled with impure hydrogen from a Kipp's apparatus, then exhausted to less than 2 cm. pressure by a water pump and re-filled with hydrogen formed by the electrolysis of water. The gas was not dried, and hence water vapour was mixed with it. It was exposed for two minutes in the bulb and then drawn into the cloud chamber and there expanded to a difference of pressure of 11·95 cm. Only a small shower resulted, while with dust-free air treated in the same way a thin fog was obtained. Water vapour alone was then tried. A little distilled water was put into the bulb, which was then exhausted by a very efficient water pump. The remaining water vapour was exposed to light for two minutes, and any nuclei formed were swept into the cloud chamber by passing a current of dust-free air through the bulb. The pressure of the water vapour was 1·55 cm. On expansion with a difference of pressure of 8·05 cm. only a small shower resulted, showing that few nuclei had been formed.

The last gas to be tried was nitrogen, which was obtained from a cylinder of nitrogen. It was tested first by sparking over caustic potash in an eudiometer and found to be very pure. Nevertheless, there was sufficient oxygen for nuclei to be produced in profusion by the ultra-violet light. The cloud chamber itself was filled with the gas, and hence it was saturated with water vapour. In order to rid it of the remaining traces of oxygen it was sparked inside the cloud chamber for 20 minutes between platinum points. Four small vessels of caustic potash stood on the base of the cloud chamber, and the cylinder in which the piston worked contained pure distilled water. Moist filter papers were placed on the metal parts of the cloud chamber to keep the space well saturated.

The table shows the result of exposing moist nitrogen, freed from any trace of oxygen, to the ultra-violet light. From it we see that no nuclei were produced, but when some air was

let in, everything else remaining the same, an exposure of five seconds produced many nuclei.

Gas.	Time of exposure.	Difference of pressure on expansion.	Cloud.
N before sparking ...	1 min.	6·95 cm.	Thin fog
N after „ ...	1 „	6·75 „	No drops
„ „ „ ...	7 „	9·25 „	No drops
„ „ „ ...	13 „	11·55 „	One or two drops
„ „ „ ...	19 „	15·2 „	Small showers
Air let in	5 secs.	9·65 „	Small fog

In all the observations with various gases the distance of the spark from the quartz window was 0·7 cm. or less.

Lenard and Ramsauer* have shown that by removing all traces of such substances as H_2O, NH_3, CO_2 from air or oxygen no nuclei which would affect a steam jet were formed by ultra-violet light. They entirely removed these substances by passing the gas through a vessel surrounded with liquid air, and thus condensing them out. The walls of the connecting tubes and the vessel in which the gas was exposed to the light had also to be very scrupulously cleansed by heating before the gas could be entirely freed of these impurities. Unless so cleaned, the walls of the vessel were able to give off a sufficient quantity of these substances for the production of nuclei, although the gas itself had first been purified by cooling; but they found that unless oxygen or CO_2 were also present no nuclei were formed.

The results of the experiments which have just been described agree with those of Lenard and Ramsauer in showing that, unless oxygen or CO_2 are present, ultra-violet light produces no nuclei. The small traces of necessary impurities must in each case have been present, but they alone are not sufficient for the production of nuclei.

§ 4. It has been suggested that ultra-violet light acting on moist air might result in the formation of particles of H_2O_2, which, by dissolving in the small drops of water, would help them to grow larger and become stable.†

An attempt was made to test this point directly by a chemical method. The quantity of H_2O_2 formed would certainly be very small, but there are very delicate tests for it; and a preliminary experiment made with actual particles of H_2O_2

* Lenard and Ramsauer, Heidelberg. "Berichte," 1910 and 1911.

† J. J. Thomson, "Conduction of Electricity Through Gases," p. 140.

showed that it was possible to detect their presence in a cloud. Titanic acid, which is ordinarily colourless, becomes yellow when a very small trace of H_2O_2 is added to it, and this was used as a detector.

First of all the cloud chamber was set up with water and dust-free air inside, so that unless the expansion was large enough to catch ions, no drops were formed. Then a small quantity of solution of H_2O_2 was let into the cloud vessel through one of the tubes in the base of it. No ultra-violet light was used, but a few drops were always obtained on expansion. The number of drops depended on the time between successive expansion. When they were made as fast as possible one after the other a few drops were formed just over the solution. If time were allowed, the particles diffused to other parts of the vessel. These nuclei must have been particles of H_2O_2, perhaps already dissolved in minute drops of water; but they cannot have been larger than the ultra-violet nuclei, for in all cases an expansion was required to form condensation on them.

The table gives some of the observations :—

Time for diffusion.	Difference of pressure.	Cloud.
A few seconds	3·5 cm.	Few drops just above solution
5 minutes	3·5 „	Very small shower near solution
30 minutes	3·1 „	Small shower in whole vessel

The presence of H_2O_2 in these showers was easily detected with titanic acid.

The apparatus was cleaned and the base of the cloud vessel covered with filter paper well soaked in solution of H_2O_2. It was then set up again with water and air, and a small open glass dish containing 0·44 gramme of titanic acid was placed in the cloud vessel in such a position that part of the clouds would fall into it.

Expansions were made with a difference of pressure of about 9·35 cm., and each time a small shower was produced, part of which fell into the acid. After 19 expansions, taking altogether half an hour, the acid appeared slightly but distinctly yellow, owing to particles of H_2O_2 having fallen into it with the showers.

The apparatus was then thoroughly cleared of H_2O_2 and

filled with dust-free air and water, about the same quantity of titanic acid having been placed inside the cloud vessel. The air was exposed to light from the spark, which was 0·7 cm. from the quartz plate.

One hundred and fifty expansions were made of 11·5 cm. difference of pressure, and the clouds produced were very much larger than the small showers on the H_2O_2 particles. They were allowed to fall into the acid, but no colouring of it could be observed, although the nuclei were as effective in producing condensation as the H_2O_2 particles—that is, a cloud could be formed on them with quite as small an expansion as on the H_2O_2 particles.

The amount of H_2O_2 formed by the action of the light must therefore, at the most, be only a small fraction of the whole change which occurs in nuclei formation.

§ 5. That ozone is produced by the action of ultra-violet light on air and oxygen has been demonstrated by many observers (recently by Pring, Proc. Roy. Soc., May 1, 1914). Lenard and Ramsauer* obtained evidence of ozone in oxygen which had been exposed to the action of light, and which beforehand had been freed from impurities by cooling; but they found that there were no condensation nuclei. If the small traces of impurities were not removed nuclei were formed in abundance. Ozone, together with these impurities, might, therefore, be one of the means by which the nuclei are formed. Experiments bearing on this point were carried out as follows :—

Oxygen from a cylinder was dried by bubbling through H_2SO_4, and was passed into a silent discharge ozoniser, and from it into the cloud chamber, which had been partially exhausted of air. On entering the cloud chamber condensation occurred at once in the ozonised oxygen, unless the amount of ozone was very small, in which case an expansion was required. The quantity of ozone was roughly estimated by the time of the discharge, and the table gives the nuclei formation resulting from the production of ozone.

Time of discharge.	Difference of pressure.	Cloud.
1–1½ minutes	No expansion	Extremely dense fog
5 seconds	,, ,,	Fog
⅗ seconds	,, ,,	Slight condensation
One make and break of coil causing dischage	9·35 cm.	Fog
Smaller quantity still	2·75 cm.	Small shower

* Lenard and Ramsauer, Heidelberg " Berichte," 1910 and 1911.

In the first observation a strip of potassium iodide starch paper placed in the cloud vessel became dark blue ; in the second observation a similar strip became slightly coloured ; while in the other observations, and about five similar ones, a strip of paper which was present all the time was not coloured at the end, although dense fogs were produced on expansion.

The formation of ozone in oxygen by a silent discharge results also in the formation of condensation nuclei. These nuclei were not only active in causing condensation of water vapour, but also of absolute alcohol, toluol and turpentine vapours. This was demonstrated by observations similar to the above ones, except that the various liquids were in turn put in the expansion apparatus.

§ 6. Experiments were then made to determine whether the nuclei would be destroyed by heating the air containing them. Two quartz tubes about 15 cm. long and 1 cm. in diameter were joined in series between the bulb and the cloud chamber, and each tube was heated for some hours by a Bunsen flame over a length of about 4 cm. in the centre. Dust-free air which had been exposed to ultra-violet light while it was in the bulb, was drawn slowly through the tubes into the cloud chamber and there expanded. A comparison was made with ozonised oxygen, and in both case the nuclei were destroyed.

From the table it is seen that dust-free air drawn slowly through the hot tubes contains practically no nuclei. Also a stream of dust-free air, exposed to ultra-violet light, while it passed through the bulb, contained nuclei sufficient to form a fog if it passed through the cold tubes, but scarcely any if it passed through the hot tubes. Similar results were obtained with the nuclei in ozonised oxygen.

Gas.	Tubes.	Time of discharge or light.	Difference of pressure.	Cloud.
Air	Hot	None	11·6 cm.	Very small shower
Air	Hot	$4\frac{1}{2}$ mins.	11·5 „	Small shower
Air	Cold	$4\frac{1}{4}$ mins.	11·6 „	Fog
Oxygen and ozone	Hot	$\frac{2}{5}$ sec.	10·25 „	Very small shower
Oxygen and ozone	Cold	$\frac{2}{5}$ sec.	10·3 „	Fog

In making these observations a curious phenomenon was observed. The result shown in the first line of the table was obtained when the air was drawn slowly through the hot tubes. But if the air, which was originally dust free and was not exposed to ultra-violet light from the spark, was drawn faster

through the hot tubes it was found to contain a very large number of nuclei, so that a fog was produced on expansion. This effect could not be got rid of, although the tubes were heated for a long time and a stream of air was drawn many times through them. It always happened if the air was drawn through the tubes too quickly. Also, there was a comparatively sharp division between the rate of flow of air through the hot tubes when the nuclei were produced in them and the rate when those existing beforehand were destroyed in them.

It is difficult to account for these nuclei, but their existence may possibly be explained thus: Since the tubes were of quartz, the nuclei may have been produced in them by light from the Bunsen flame, which was playing round the outside of the tubes; and these nuclei may have passed away from the hot part of the tubes before they became hot enough to be destroyed. If the passage of the air through the tubes was slower, they and any other nuclei which were there would have time to be heated and destroyed.

One other property of the nuclei which was observed was that it was not possible to draw them with a stream of air through a plug of cotton wool. They were all filtered out of the air like dust particles.

Summary.

1. Nuclei produced in air by ultra-violet light which has traversed a few centimetres of air are not affected by an electric field of 50 volts per centimetre.

2. The nuclei are equally effective in producing condensation of water, toluol and turpentine vapours, and they are formed even by light which has traversed 50 cm. of air.

3. Alcohol vapour condenses without expansion on much smaller nuclei than does water vapour.

4. No nuclei were formed by the light unless oxygen or CO_2 were present in the gas.

5. No trace of H_2O_2 could be detected in the clouds formed on the nuclei.

6. Oxygen containing ozone also contains nuclei for condensation, and these nuclei have similar properties to those formed by ultra-violet light.

7. The nuclei can be destroyed by heating the air containing them.

In conclusion, it seems probable that the nuclei formed by ultra-violet light do not cause condensation by virtue of any

particular chemical composition, but that they are particles large enough to act like dust particles as centres round which condensation can begin. It is probable that some of the dust particles in the air are really nuclei formed by the action of ultra-violet light.

I should like here to express my thanks to Mr. Wilson, Dr. Womack and Dr. J. F. Spencer for the help they have so kindly given me.

ABSTRACT.

1. Nuclei produced in air by ultra-violet light which has traversed a few centimetres of air are not affected by an electric field of 50 volts per centimetre.

2. The nuclei are equally effective in producing condensation of water, toluol and turpentine vapours, and they are formed even by light which has traversed 50 cm. of air.

3. Alcohol vapour condenses without expansion on much smaller nuclei than does water vapour.

4. No nuclei were formed by the light unless oxygen or CO_2 was present in the gas.

5. No trace of H_2O_2 could be detected in the clouds formed on the nuclei.

6. Oxygen containing ozone also contains nuclei for condensation, and these nuclei have similar properties to those formed by ultra-violet light.

7. The nuclei can be destroyed by heating the air containing them.

It seems probable that the nuclei formed by ultra-violet light do not cause condensation by virtue of any particular chemical composition, but that they are particles large enough to act like dust particles as centres round which condensation can begin.

DISCUSSION.

Prof. O. W. Richardson said the author had succeeded in eliminating a number of substances to which these effects had from time to time been attributed. The method employed was extremely sensitive, one or two ions being easily detected. Of the various substances eliminated the only one which he was not quite certain of was ozone. The elimination of this seemed to rest on the results of Lenard and Ramsauer. Did the author think these results were absolutely conclusive ?

Mr. D. Owen said that the explanation of the action of ozone in causing cloud formation was difficult if regarded as a direct effect. Possibly ozone acting on water vapour might give rise to hydrogen peroxide, in which case the suggestion of Mr. C. T. R. Wilson of a lowering of the saturation vapour pressure would apply. However the failure, in the case of fogs produced by ultra-violet light, to detect hydrogen peroxide by the titanium oxide test, which, according to Dr. Senter, can detect one part of hydrogen peroxide in 10 millions, seemed to exclude the action of hydrogen peroxide : this test is, moreover, a specific test for hydrogen peroxide even in the presence of other oxidising agents. As a possible alternative, the agency of oxides of nitrogen might be suggested. It would be of interest to test whether the phenomenon occurs when nitrogen is entirely excluded. In conjunction with chemical tests of amount present, quantitative data enabling the resulting lowering of the vapour pressure to be calculated should enable the problem in question to be definitely solved.

Dr. H. Borns remarked that the detection or identification of hydrogen peroxide was not always easy. In experiments of the kind described in the Paper effects might be ascribed to hydrogen peroxide which were really due to nitrogen oxides. Such oxides would be produced when air was drawn through hot quartz tubes, and might help to account for the peculiar observations mentioned in the last part of the Paper.

Miss Saltmarsh, in reply, was not aware of any work having been done on the effect of oxides of nitrogen in producing condensation nuclei. In reply to Prof. Richardson, Lenard and Ramsauer had not stated that ozone was ineffective. They laid great stress on the presence of minute quantities of impurities. They attempted to get rid of these by condensing them out at low temperatures. They held that the ozone reacting on these traces of impurities was instrumental in producing the effects.

XXVIII. *On the Self-induction of Solenoids of Appreciable Winding Depths.* *By* S. Butterworth, *M.Sc., Lecturer in Physics, School of Technology, Manchester.*

Received February 27, 1915.

1.

The only formulæ which appear to have been given for the self-induction of solenoids in which the correction for winding depth is included are those of Rosa* and Cohen.† In arriving at these formulæ certain doubtful approximations have been made. In the present Paper formulæ have been developed which are free from such approximations,‡ and it is shown that, while Rosa's formula gives better results than that of Cohen, it does not possess the accuracy claimed for it by its author.

2. *Cohen's Method.*

The coil under consideration is divided into a finite number of layers, each of which is sufficiently thin to be treated as a cylindrical current sheet. The self-induction of the coil is then found by combining the self and mutual inductions of these layers according to the usual laws. For the self-inductions Cohen uses the formula,

$$L=4\pi^2n^2a^3\left(\frac{2+c^2}{\sqrt{4+c^2}}-\frac{8}{3\pi}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which a is the mean radius of a layer, ca the coil length, and n the number of turns per unit length.

The formula is an approximation to an exact elliptic integral formula due to Lorenz.

For the mutual inductions he uses

$$M=4\pi^2n^2a^3(\sqrt{c^2+a^2/A^2}-A/a+a/8A), \quad . \quad . \quad . \quad (2)$$

in which a and A are the radii of the inner and outer cylinder

* Rosa, "Bulletin" of the Bureau of Standards, 4, p. 369, 1908.

† Cohen, "Bulletin" of the Bureau of Standards, 4, p. 384, 1908.

‡ The formulæ of this Paper are approximate in that they neglect (*a*) the insulation space between the wires of the coil, (*b*) the effect of the helicity of the winding.

respectively, and ca is the length of the cylinders. This is an approximation to a formula due to Maxwell.*

The final formula obtained by Cohen for the thick coil is

$$L/\pi^2N^2R^3=\frac{4}{m}\left(\frac{2+c^2}{\sqrt{4+c^2}}-\frac{8}{3\pi}\right)$$

$$+\frac{8}{m^2}\left[\left\{(m-1)k_1^2+(m-2)k_2^2+\ldots\right\}\left(\sqrt{k_1^2+c^2}-\frac{8}{3\pi}k_1\right)\right.$$

$$-\frac{\delta k}{2}\left\{m(m-1)k_1^2+(m-1)(m-2)k_2^2+\ldots\right\}\left(1-\frac{k_1}{\sqrt{k_1^2+c^2}}\right)$$

$$\left.-\frac{\delta k}{16}\left\{m(m-1)k_1^2+(m-2)(m-3)k_2^2+\ldots\right\}\right] \quad . \quad . \quad . \quad (3)$$

in which m is the number of layers. N the number of turns per unit length (including all the layers), cR the coil length, k_1R, k_2R, . . . the mean radii of the layers ($k_{r+1}>k_r$), and $R\delta k$ the width of each layer.

He states that this formula has a minimum accuracy of one-half of 1 per cent. when $c>4$.

An *a priori* test of this assertion may be obtained by an examination of formulæ (1) and (2). Since the distance apart of contiguous layers is small compared with their radii, (2) should approximately agree with (1) when $A=a$, *i.e.*,

$$L=4\pi^2N^2a^3\left(\sqrt{1+c^2}-\frac{8}{3\pi}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

should not differ by more than one half of 1 per cent. from (1) when $c>4$.

Further, we may test the absolute accuracy of (1) and (2) by means of some exact formula, such as that of Havelock.† viz.,

$$L=4\pi^2N^2a^3c\left\{1-\frac{8}{3\pi c}+\frac{1}{2c^2}\right)-\frac{1\cdot3}{2\cdot3}\frac{1}{c^2}\right)-\frac{3\cdot5}{3\cdot4}\frac{1}{c^2}\right)\ldots\Big\} \quad . \quad (5)$$

in which the notation $)x$ means that any term is obtained from the preceding term by multiplying by x.‡

* Maxwell, "Electricity and Magnetism" (Vol. II., Art. 678).

† Havelock, "Phil. Mag.," Vol. XV., p. 332, 1908.

‡ By using this notation, computation from the formula is far more rapid than when each term in the series is calculated separately.

Table I. gives the values of $L/4\pi^2N^2a^3$ as obtained from the three formulæ.

TABLE I.

c.	Havelock.	Formula (1).	Formula (4).
4	3·2725430	3·1761	3·2481
5	4·2492672	4·1650	4·2240
6	5·2333879	5·1596	5·2078

The values from Havelock's formula are calculated to eight figures as they are needed later. The table shows that neither of the fundamental formulæ used by Cohen are sufficiently accurate to give results to one-half of 1 per cent.

3. *Rosa's Method.*

The inductance (L_0) of a cylindrical current sheet of the mean radius of the thick coil is calculated by any suitable exact formula, and the correction for thickness is got by deducting from L_0 an amount ΔL, where

$$\frac{\Delta L}{\pi^2N^2R^3}=\frac{8}{\pi}cT(A+B), \quad \ldots \ldots \quad (6)$$

in which N, R, c have the same meanings as before, 2TR is the winding depth, and A and B are tabulated functions of T and c/T respectively. The following values of A and B taken from Rosa's tables are required later :—

TABLE II.

c.	T.	A.	B.
4	1/10	0·6922	0·3099
5	1/12	0·6926	0·3218
6	1/10	0·6922	0·3218

The method of obtaining A and B is to divide the coil into slices by planes perpendicular to the axis, such that each slice forms a coil with a winding channel of square section. The correction A arises from the difference between the self-induction of a coil of square section and that of a current sheet of the mean radius of the coil. Weinstein's formula is used for the square-sectioned coil and Rayleigh's formula for the current sheet. Apart from the question of the validity of Weinstein's formula up to $2T=0·25$ (the maximum thickness for which Rosa tabulates A), there appears to be no serious error in this correction.

The correction B is due to the mutual induction between the slices. It is obtained by the method of geometrical mean

distance, and herein lies the most probable source of error, as the method of geometrical mean distance is only valid for coils whose sectional dimensions *and distance apart* are small compared with their mean radii. The latter condition is far from being the case with coils of the nature under consideration. It will be shown in Section 10 that Rosa's correction is 2 per cent. in error for coils whose length is four times their radius, while the error in the total induction is 0·2 per cent.

4. *Present Method.*

The coil is first taken to be a portion of an infinite coil, and the self-induction (L_1) calculated under these conditions. The work to be done to remove the portion from the remainder of the infinite coil is then determined, and this gives the diminution in self-induction due to the effect of the pole faces of the coil. This work is due to the attraction of two pairs of unlike poles in contact, together with the repulsion of two pairs of like poles at a distance apart equal to the length of the coil, or, since the work to be done to separate a pair of poles is equivalent to the mutual induction between the corresponding semi-infinite coils, the end correction to be deducted from L_1 is

$$2M(o)-2M(l),$$

where $M(l)$ represents the mutual induction between two semi-infinite coils at a distance, l.

Hence, if ΔL_1 represents the (additive) thickness correction for L_1, $\Delta M(l)$ that for $M(l)$, the total thickness correction is

$$\Delta L=\Delta L_1-2\Delta M(o)+2\Delta M(l) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

5. *The Correction* ΔL_1.

The self-induction of a length, l, of an infinite coil of outer and inner radii a and b is given by Maxwell* as

$$L_1=\frac{2}{3}\pi^2N^2l(a^2+2ab+b^2),$$

where N is the number of turns per unit length. Hence, if $a=R(1+T)$, $b=R(1-T)$, $l=Rc$

$$L_1=4\pi^2N^2R^3c\left(1-\frac{2}{3}T+\frac{1}{3}T^2\right),$$

so that the thickness correction ΔL_1 is given by

$$\frac{\Delta L_1}{\pi^2N^2R^3}=-\frac{8}{3}cT\left(1-\frac{1}{2}T\right) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

* Maxwell, "Electricity and Magnetism," Vol. II., Art. 679.

6. *The Correction* $\Delta M(o)$.

The mutual induction between two semi-infinite, coaxial, cylindrical current sheets with coplanar ends is

$$m=\frac{8}{3}\pi a\{(a^2+b^2)E-(a^2-b^2)K\} \quad . \quad . \quad . \quad . \quad (9)$$

where a and b are the radii of the two sheets ($a>b$) and K and E are complete elliptic integrals of the first and second kinds to modulus b/a. When b/a is nearly unity, K and E are suitably expressed by the series

$$\left.\begin{aligned} K=&\log\frac{4}{k}+\frac{1^2}{2^2}k^2\left(\log\frac{4}{k}-\frac{2}{1\cdot2}\right)\\ &+\frac{1^2\cdot3^2}{2^2\cdot4^2}k^4\left(\log\frac{4}{k}-\frac{2}{1\cdot2}-\frac{2}{3\cdot4}\right)\cdot\cdot\cdot\cdot\cdot\\ &+\cdot\cdot\cdot\\ E=\ &1+\frac{1}{2}k^2\left(\log\frac{4}{k}-\frac{1}{1\cdot2}\right)\\ &+\frac{1^2}{2^2}\cdot\frac{3}{4}k^4\left(\log\frac{4}{k}-\frac{2}{1\cdot2}-\frac{1}{3\cdot4}\right)\cdot\cdot\cdot\cdot\cdot\\ &+\cdot\cdot\cdot \end{aligned}\right\} \quad . \quad . \quad (10)$$

in which $k^2=1-\frac{b^2}{a^2}$.

If we put $a=r+x$, $b=r-x$, so that

$$k^2=\frac{4xr}{(x+r)^2}=4\frac{x}{r}\left(1-2\frac{x}{r}+3\frac{x^2}{r^2}-4\frac{x^3}{r^3}+\cdot\cdot\cdot\right)$$

(10) becomes, on application of the usual logarithmic and binominal expansions,

$$\left.\begin{aligned} K=&\frac{1}{2}\log\frac{4r}{x}\left(1+\frac{x}{r}+\frac{1}{4}\frac{x^2}{r^2}+\frac{1}{4}\frac{x^3}{r^3}+\cdot\cdot\cdot\right)\\ &-\frac{1}{8}\frac{x^2}{r^2}-\frac{1}{8}\frac{x^3}{r^3}+\cdot\cdot\cdot\\ E=&\frac{1}{2}\log\frac{4r}{x}\left(2\frac{x}{r}-\frac{x^2}{r^2}+\frac{3}{2}\cdot\frac{x^3}{r^3}-\frac{9}{8}\frac{x^4}{r^4}+\cdot\cdot\cdot\right)\\ &+1-\frac{x}{r}+\frac{3}{4}\frac{x^2}{r^2}-\frac{x^3}{r^3}+\frac{153}{192}\frac{x^4}{r^4}+\cdot\cdot\cdot \end{aligned}\right\} \quad . \quad . \quad (11)$$

Applying (11) to (9)

$$m=\frac{16}{3}\pi r^3\left\{1+\frac{3}{4}\frac{x^2}{r^2}+\frac{3}{64}\frac{x^4}{r^4}+\ \ldots\right.$$
$$\left.-\frac{3}{2}\log\frac{4r}{x}\left(\frac{x^2}{r^2}-\frac{1}{8}\frac{x^4}{r^4}+\ldots\right)\right\}\quad\ldots\quad(12)$$

In order to find the mutual induction between two semi-infinite coils, A and B, in contact, each having mean radius R, winding depth 2 X and n turns per unit area of section, divide the coils into cylindrical filaments of width δl and add the mutual inductions of the filaments as follows :—

Take the mutual induction between a filament of radius $r+x$ in coil A and one of radius $r-x$ in coil B. Associate with it the (same) mutual induction between the filament $r-x$ in A and $r+x$ in B. For these two pairs the sum of the mutual inductions is

$$\delta\mathrm{M}=2n^2m(\delta l)^2\quad\ldots\quad\ldots\quad(13)$$

Let r vary by steps $\delta r=\delta l$ from $r=\mathrm{R}-\mathrm{X}-x$ to $r=\mathrm{R}-\mathrm{X}-x$ The sum of the expressions $\delta\mathrm{M}$ will then give the sum of the mutual inductions of all filaments at a distance $2x$ apart.

Perform a second summation with x varying by steps $\delta x=\frac{1}{2}\delta l$ from $x=0$ to $x=\mathrm{X}$. This will give the required mutual induction $\mathrm{M}(o)$. Hence

$$\mathrm{M}(o)=4n^2\int_0^{\mathrm{X}}dx\int_{\mathrm{R}-\mathrm{X}+x}^{\mathrm{R}+\mathrm{X}-x}m\,dr\quad\ldots\quad(14)$$

Applying (14) to (12)

$$\mathrm{M}(o)=\frac{64}{3}n^2\pi\mathrm{R}^3\mathrm{X}^2\left\{1+\frac{23}{48}\frac{\mathrm{X}^2}{\mathrm{R}^2}-\frac{1}{1600}\frac{\mathrm{X}^4}{\mathrm{R}^4}\right.$$
$$\left.-\log\frac{4\mathrm{R}}{\mathrm{X}}\left(\frac{1}{4}\frac{\mathrm{X}^2}{\mathrm{R}^2}-\frac{1}{80}\frac{\mathrm{X}^4}{\mathrm{R}^4}\right)\right\}\quad\ldots\quad(15)$$

or putting $\mathrm{X}=\mathrm{TR}$, $2n\mathrm{X}=\mathrm{N}$, so that N is the number of turns per unit length in either coil, the correction $\Delta\mathrm{M}(o)$ is given by

$$\frac{\Delta\mathrm{M}(o)}{\pi^2\mathrm{N}^2\mathrm{R}^3}=-\frac{4}{3\pi}\left\{\mathrm{T}^2\left(\log\frac{4}{\mathrm{T}}-\frac{23}{12}\right)-\frac{\mathrm{T}^4}{20}\left(\log\frac{4}{\mathrm{T}}-\frac{1}{20}\right)\right\}.\quad(16)$$

7.

In order to see whether the effect of higher powers of T is appreciable, the mutual induction $\mathrm{M}(o)$ will now be found by

another method. In a Paper on thick coaxial coils* the author has shown that the mutual induction $N(o)$ between two semi-infinite coils of outer radii unity and r respectively, of zero inner radius, and with their ends in contact, is given by

$$\frac{N(o)}{2\pi^2 r^3}=\frac{1}{6}\left\{1-\frac{3}{20}r^2\left(\log\frac{4}{r}+\frac{9}{20}\right)+3r^4\sigma\right\}, \quad . \quad . \quad (17)$$

where $\sigma=\sum_0^\infty\left(\frac{1\cdot3\ldots2n+3}{2\cdot4\ldots2n+4}\right)^2\frac{r^{2n}}{(2n+7)(2n+3)(n+3)(n+1)}$, (17A)

the winding density of each coil being unity. When $r=1$ we have the alternative form

$$\frac{N_1(o)}{2\pi^2}=\frac{1}{30\pi}\left\{17-6\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots\right)\right\}$$

$$=0\cdot122063419. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Further, the mutual induction $M(o)$ between two similar hollow coils of external and internal radii unity and r is

$$M(o)=(1+r^5)N_1(o)-2N(o). \quad . \quad . \quad . \quad . \quad (19)$$

Since in (17) the general term is given, any desired accuracy may be obtained by calculating a sufficient number of terms, although when r is nearly equal to unity the method becomes tedious, as (19) is then the difference between two nearly equal quantities.

Table III. compares the values of $M(o)$ as found by (17) and (19), and by (16). The second column is calculated by (17), the third by (19) and the fourth by (16). It is seen that the value of $N_1(o)$ is in good agreement with (18) and that formula (16) is extremely accurate for all winding depths less than one-quarter the radius of the coils.

TABLE III.

r.	$N(o)/2\pi^2r^3$.	$M(o)/2\pi^2$.	
		By (19).	By (16).
1·00	0·122063423	...	...
0·95	0·125119075	0·001965839	0·0019658390
0·90	0·12817228	0·007265462	0·0072654615
0·85	0·13120878	0·01506643	0·015066436
0·80	0·13421539	0·02462460	0·02462459

* Butterworth, "Phil. Mag.," p. 578, April, 1915.

8. *The Correction* $\Delta M(l)$.

It is shown in the Paper referred to in the preceding section that the mutual induction between two semi-infinite coaxial coils of outer radii unity and r, inner radius zero and at distance z apart is given by

$$N(z,r)=\frac{\pi^2 r^3}{z}\left\{\frac{1}{9}-\frac{1}{(2z)^2}\left(\frac{1}{15}+\frac{1}{15}r^2\right)\right.$$
$$+\frac{1}{(2z)^4}\left(\frac{2}{21}+\frac{6}{25}r^2+\frac{2}{21}r^4\right)$$
$$-\frac{1}{(2z)^6}\left(\frac{5}{27}+\frac{6}{7}r^2+\frac{6}{7}r^4+\frac{5}{27}r^6\right)$$
$$\left.+\ldots\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

the winding density of each coil being unity and z being greater than $2r$.

From the laws of combination of mutual inductances and from dimensional considerations, the mutual induction between two similar hollow coils of outer and inner radii a and b, and at a distance c, is

$$M(c)=a^5\left\{N(z,1)-2N(z,r)+r^5N\left(\frac{z}{r},1\right)\right\}, \quad . \quad . \quad (21)$$

where $z=c/a$, $r=b/a$.

The relation (21) when applied to (20) converts the term r^{2n+3}/z^{2p+1} into $\frac{a^5}{z^{2p+1}}(1-2r^{2n+3}+r^{2p+6})$ and the term $r^{2p-2n+3}/z^{2p+1}$ into $\frac{a^5}{z^{2p+1}}(1-2r^{2p-2n+3}+r^{2p+6})$.

Since these terms have the same coefficient their sum is converted into

$$\frac{2a^5}{z^{2p+1}}(1-r^{2n+3})(1-r^{2p-2n+3})$$

or
$$\frac{2}{c^{2p+1}}(a^{2n+3}-b^{2n+3})(a^{2p-2n+3}-b^{2p-2n+3}). \quad . \quad (22)$$

Hence, if in (20) we replace $2r^{2n+3}/z^{2p+1}$ by the expression (22) we obtain the required series for $M(c)$. Making this

substitution, and in addition putting $a=1+T$, $b=1-T$, we find on arranging in ascending powers of T

$$\frac{cM(c)}{4\pi^2}=T^2\left(1-\frac{1}{2c^2}+\frac{5}{8c^4}-\frac{35}{32c^6}+\dots\right)$$
$$+\frac{2}{3}T^4\left(1-\frac{7}{4c^2}+\frac{17}{4c^4}-\frac{775}{64c^6}+\dots\right)$$
$$+\frac{1}{9}T^6\left(1-\frac{39}{10c^2}+\frac{507}{20c^4}-\frac{2167}{80c^6}+\dots\right)$$
$$+\dots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

which holds for coils of mean radius unity, winding depth 2T, length c, and unit winding density. By multiplying $M(c)$ by n^2R^3 we get the formula for coils of mean radius R, winding depth 2TR, length cR, and winding density n. Finally, since $2n$TR=N, the number of turns per unit length

$$\frac{\Delta M(l)}{\pi^2N^2R^3}=\frac{2}{3}\frac{T^2}{c}\left(1-\frac{7}{4c^2}+\frac{17}{4c^4}-\frac{775}{64c^6}+\dots\right)$$
$$+\frac{1}{9}\frac{T^4}{c}\left(1-\frac{39}{10c^2}+\dots\right)$$
$$+\dots. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

9.

As a check on (24), the value of $M(l)$ can be calculated by (24) and again by (20) and (21) directly. In getting $M(l)$ for the mean radius the formula

$$M(l)=\frac{\pi^2N^2R^3}{c}\left\{1-\frac{1}{2c^2}\right)-\frac{3\cdot5}{3\cdot4}\frac{1}{c^2}\right)-\frac{5\cdot7}{4\cdot5}\frac{1}{c^2}\right)\dots\Big\}, \quad . \quad (25)^*$$

due to Havelock, may be used. The first term in (23) agrees with (25).

For example, with R=9, 2TR=2, cR=40, N=200,

(20) and (21) give $M=6\cdot454_3\pi^2$ millihenries.

(24) and (25) give $M=6\cdot4544_1\pi^2$ millihenries.

The agreement is satisfactory within the limits of accuracy of the formulæ.

* *See* footnote on p. 372.

10. *Comparison of the Corrections.*

Table IV. shows the results obtained by Rosa's, Cohen's and the present method for three coils, for which

$$c=4,\ 5,\ 6,\ ;\ T=1/10,\ 1/12,\ 1/10,\ \pi^2N^2R^3=1.$$

It is seen that Rosa's correction is 2 per cent. too high, while the total induction is one-fifth per cent. low.

In Cohen's method the tabulated values are for five layers. Although the results show very good agreement with the correct values, this is accidental, being due to the number of layers chosen. If a larger number of layers are taken the results get worse instead of better. Thus, for $c=4$, $T=1/10$, with

$m=$	1	2	3	4	5	10	infinity
L=	12·70,	12·11,	12·07,	12·06,	12·09,	12·14,	12·19

the correct value being 12·09.

TABLE IV.

c	4	5	6
T	1/10	1/12	1/10
ΔL_1 by (8)	−1·0133333	−1·0690236	−1·5200000
$\Delta M(o)$ by (16) ...	−0·0075138	−0·0057568	−0·0075138
$\Delta M(l)$ by (24) ...	+0·0015093	+0·0008676	+0·0010621
ΔL by (7)	−0·995287	−1·055775	−1·502848
ΔL by (6), Rosa...	−1·0207	−1·0763	−1·5492
L_0 by (5)	13·090172	16·997069	20·933552
L present method)	12·094885	15·941294	19·430704
L (Rosa)	12·0695	15·9208	19·3843
L (Cohen) by (3) using 5 layers ...	12·091	15·939	19·462

11.

It is convenient to summarise the steps by which the inductance of a solenoid can be calculated. The coil is assumed to have a length greater than twice its diameter and a thickness of winding of less than one-tenth the diameter. The formulæ are sufficient to give four figure accuracy in the final result. The method is as follows :—

1. Calculate the inductance for the mean radius R without the end correction from the formula,

$$L_0=4\pi^2N^2R^3c,$$

where N is the number of turns per unit length (including all layers) and Rc is the coil length.

2. Apply the "end" correction to obtain the inductance

(L_2) for a current sheet of the mean radius of the coil from the formula,

$$L_2=L_1\left(1-\frac{8}{3\pi c}+\frac{1}{2c^2}-\frac{1}{4c^4}\right).$$

3. Apply the "thickness" correction to obtain the inductance (L_3) for a coil of winding depth 2TR from the formula,

$$L_3=L_2+\Delta L_2,$$

in which

$$\Delta L_2=-\frac{2}{3}L_1T^2\left\{1-\frac{T}{2}-\frac{T}{\pi c}\left(\log_e\frac{4}{T}-\frac{23}{12}\right)\right\}.$$

4. Apply the "insulation space" correction to obtain the true inductance L, for which *see* Rosa ("Bull." Bureau of Standards, 3, p. 37, 1907).

12. *Summary.*

1. It has been shown that the formulæ previously given for correcting for thickness in determining the inductance of solenoids fail to give an accuracy of one part in a thousand.

2. The true correction formulæ are obtained which are capable of giving eight figure accuracy without undue labour.

3. Simplified formulæ are also given which hold when only four figure accuracy is required.

APPENDIX.

As portions of the criticism of Rosa's and Cohen's formulæ depend on arithmetical accuracy in calculation, the arithmetical details for $c=4$, $T=1/10$ are appended.

(*a*) *Inductance of Mean Layer Solenoid.*

By Havelock's formula (5):

$$\begin{aligned}L_0=16\{&1-0\cdot21220659+0\cdot03125000\\&-\quad97656+\quad7630\\&-\quad835+\quad109\\&-\quad16+\quad3\}\end{aligned}$$

$$\begin{aligned}&=16\{1\cdot03132742\\&\quad-0\cdot21319166\}\\&=16\times0\cdot81813576\\&=13\cdot090172.\end{aligned}$$

(*b*) *Correction* ΔL_1 (formula (8)).

$$\Delta L_1=-\frac{8}{3}\times4\times\frac{1}{10}(1-0\cdot05)=-\frac{30\cdot4}{30}=+1\cdot0133333.$$

(*c*) *Correction* $\Delta M(o)$ (formula (16)).

$$\Delta M(o)=-\frac{4}{3\pi}\{0\cdot01(3\cdot68888-1\cdot91667)-0\cdot000005(3\cdot69-0\cdot05)\}$$

$$=-\frac{4}{3\pi}\{0\cdot0177221-0\cdot0000182\}$$

$$=-0\cdot424413\times0\cdot0177039$$

$$=-0\cdot00751376.$$

(*d*) *Correction* $\Delta M(l)$ (formula (24)).

$$\Delta M(l)=\begin{Bmatrix}0\cdot0016667-0\cdot0001823\\+\quad 277-\quad 49\end{Bmatrix}+\begin{Bmatrix}0\cdot0000028\\-\quad 7\end{Bmatrix}$$

$$=0\cdot0015093.$$

(*e*) *Correction* ΔL (formula (7)).

$$\Delta L=\Delta L_1-2\Delta M(o)-2\Delta M(l)$$

$$=-1\cdot0133333+0\cdot0150275-0\cdot0030186$$

$$=-0\cdot9952872.$$

(*f*) *Rosa's Correction* (formula (6) and Table II.).

$$\Delta L=-\frac{8}{\pi}\times\frac{4}{10}(0\cdot6922+0\cdot3099)$$

$$=-0\cdot31831\times3\cdot2\times1\cdot0021$$

$$=-1\cdot0207.$$

(*g*) *Cohen's Formula* (3).

Using five layers.

$m=5.$ $\delta k=0\cdot04.$

$k_1=0\cdot92$	$k_2=0\cdot96$	$k_3=1\cdot00$	$k_4=1\cdot04$
$k_1^2=0\cdot8464$	$k_2^2=0\cdot9216$	$k_3^2=1\cdot0000$	$k_4^2=1\cdot0816$
$(m-1)k_1^2=3\cdot3856$	$m(m-1)k_1^2=16\cdot9$		$m(m-1)k_1^2=16\cdot9$
$(m-2)k_2^2=2\cdot7648$	$(m-1)(m-2)k_2^2=11\cdot1$		$(m-2)(m-3)k_2^2=5\cdot5$
$(m-3)k_3^2=2\cdot0000$	$(m-2)(m-3)k_3^2=6\cdot0$		
$(m-4)k_4^2=1\cdot0816$	$(m-3)(m-4)k_4^2=2\cdot2$		
Sum $=9\cdot2320$	Sum $=36\cdot2$		Sum $=22\cdot4$

$$\frac{2+c^2}{\sqrt{4+c^2}}-\frac{8}{3\pi}=0\cdot9\sqrt{20}-\frac{8}{3\pi}=\begin{cases}4\cdot0249\\-0\cdot8488\end{cases}=3\cdot1761$$

$$\sqrt{k_1^2+c^2}-\frac{7}{8}k_1=\sqrt{16\cdot846}-0\cdot805=\begin{cases}4\cdot1045\\-0\cdot8050\end{cases}=3\cdot2995$$

$$1-\frac{k_1}{\sqrt{k_1^2+c^2}}=\frac{3\cdot1845}{4\cdot1045}=0\cdot775.$$

Substituting in formula (3)

$$L=\frac{4}{5}\times 3{\cdot}1761+\frac{8}{25}(9{\cdot}2320\times 3{\cdot}2995 \\ -0{\cdot}02\times 36{\cdot}2\times 0{\cdot}775-0{\cdot}056)$$

$$=2{\cdot}5409+0{\cdot}32(30{\cdot}461-0{\cdot}617)$$

$$=2{\cdot}5409+9{\cdot}5501=12{\cdot}0910.$$

ABSTRACT.

The existing formulæ for coils of this type—viz., those of Rosa and Cohen—are shown to be inaccurate, the error amounting to one-fifth of 1 per cent. for the best formula when the winding depth is one-tenth the diameter of the coil. For greater winding depths the error is larger. The inaccuracy in Rosa's formula is due to the neglect of curvature in correcting for thickness, while in Cohen's formula the error is due to the approximate method of development.

New formulæ are developed by methods which are free from such approximations, and which apply to any coil for which the length is greater than twice the diameter, and the winding depth is less than one-tenth the diameter. These formulæ are capable of giving eight-figure accuracy.

Simplified formulæ are also given which are suitable when only four-figure accuracy is required.

XXIX. *Precision Resistance Measurements with Simple Apparatus.* *By* E. H. Rayner, *M.A., National Physical Laboratory.*

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The object of this Paper is to give some hints as to the comparison of resistances to a higher degree of accuracy than is usually attempted (except in the case of the inter-comparison of standards and of platinum thermometry), but which is often required for the determination of the constancy and of the temperature coefficient of resistance alloys and in the accurate estimation of temperature by resistance methods.

Simple resistance thermometers of copper or iron, &c., ought to be capable of a sensitivity of 0·001° C. This implies measuring resistance changes of about 1 in 300,000. Where very small temperature differences are in question, such as the lowering of the freezing point of solutions, there would appear to be no difficulty in making up a simple Wheatstone bridge with one pair of opposite sides of copper and the other of manganin. The whole might with advantage be immersed in the liquid whose temperature is to be measured. This would avoid the troublesome corrections commonly required in resistance thermometry. Balance would be obtained by a high resistance shunt in parallel with one side of the quadrilateral. Such an arrangement ought to be able to detect changes of temperature of the order of a ten-thousandth of a degree, implying a corresponding change of 1 part in 2,000,000 or 3,000,000 in the resistance.

It is known that small variations in the composition of alloys very often materially alters not only their resistance, but also the rate of change of their resistance with temperature. An accurate determination of the temperature coefficient affords a very delicate criterion of the chemical and physical similarity of such alloys, which will in many cases be much more sensitive and quickly performed than any ordinary chemical analysis.

The addition of another significant figure to the accuracy attainable in any physical measurement has always provided a new tool for use in research, and in measurements of resistances high accuracy is particularly easy of attainment, and the results may often be correlated with other physical and chemical properties.

Measurements of this nature are often neglected, owing to the idea that specially accurate Wheatstone, Carey-Foster or Kelvin bridges are required. This is by no means the case, and the results of some little experience in precision resistance measurements with simple apparatus may be useful to others.

The subject has been dealt with by R. T. Glazebrook in various British Association Reports, among which may be mentioned that of 1883, and in an appendix to the Report of 1906 F. E. Smith gives a valuable discussion of the subject. In this Paper Mr. Smith deals with precision resistance measurements more in connection with the comparison of the standards under his charge than with regard to practical hints as to everyday work. Various publications of the Bureau of Standards Washington and of the Reichsanstalt in Instrumentenkunde have drawn attention to the most likely sources of error, especially in the measurement of low resistances. Some of these are given in the bibliography at the end of this Paper.

The use of Bridge methods only is discussed. Potentiometer methods requiring great steadiness of current, which are sometimes of considerable magnitude, require equipment not commonly available.

Though one can hardly claim much novelty in the principles underlying resistance measurements, little seems to have been written on the application of precision methods to other than the inter-comparison of standard resistances. From the number of inquiries on the subject a description of methods suitable to various circumstances ought to be useful.

Many electrical indicating instruments can be read to 1 part in 1,000, and this indirectly fixes the lower limit of accuracy of resistances which can be of any practical use. A resistance box with errors of 1 part in 1,000 would, in fact, be a continual trouble, and a few parts in 10,000 may be regarded as a reasonable limit for Wheatstone bridge and similar resistances, and 1 or 2 parts in 10,000 for standard resistances. Precision measurements may be conveniently defined as those in which an accuracy of 1 part in 10,000 or more is attained.

The measurements described in this Paper attain the accuracy of 1 in 10,000 for practically every case, and in favourable conditions 1 in 1,000,000, or more. The ordinary Wheatstone bridge with plug contacts is not suitable for measurements of this nature, as it does not allow of a sufficiently continuous variation in resistance. If further accuracy is attempted in the usual manner by increasing the ratio between

the bridge arms the arrangement becomes insensitive. As an adjunct to a number of other resistances, it, or some similar resistance variable by 0·1 ohm up to 10,000 ohms, will be essential.

In measurements of this nature it is much easier to follow the methods adopted and accuracy attainable if numerical examples are given, and the results of experimental measurements are reproduced for this purpose. These have all been made with a Broca galvanometer of 10 ohms resistance. One of 100 ohms would have been more suitable for practically every experiment quoted, and for many one of 1,000 ohms, so that considerable advantage can be obtained in this respect by choice of one of a more suitable resistance. Still, the Paper shows what can be done when somewhat handicapped in this way. Nature has kindly allowed, when using a galvanometer of resistance R on a Wheatstone bridge instead of one of the most suitable resistance nR, that the sensitivity varies very slowly for a considerable change in n.

Thus, as the galvanometer has a resistance 25 times as high or as low as the most suitable value the sensitivity is only reduced to 0·4 of the best obtainable.*

For satisfactory work a voltmeter or ammeter to measure the main current is essential. In all measurements the galvanometer circuit should be kept closed, and the current to the bridge reversed, and, for quick working, resistances should be non-inductively wound. If reversing gives a large kick due to inductivity, the galvanometer circuit may be opened momentarily to reduce the effect. It is necessary that the galvanometer should not be affected electrostatically, and arrangements may be necessary to prevent this trouble. If the galvanometer is of the moving coil type it may be found necessary to connect the magnet and case to one terminal of the instrument, and if necessary to insulate the whole instrument separately if it has not insulating feet.

If the galvanometer is of the moving magnet type it is desirable that there should be electrical conductivity from the magnets and damping vane through the suspension to one of the terminals in a similar manner. This necessitates silvering the suspension if of quartz or glass. The conductivity of a silk suspension, if not specially cleaned but left "dirty," seems to be sufficient to avoid the trouble.

* Schuster, *see* Bibliography.

A metallic shield such as tinfoil between the galvanometer coils and the moving part may be used in place of a conducting suspension. This shield, and the portion of the instrument supporting the moving system, should be at some fixed, such as "earth," potential. The point to bear in mind is that on suddenly applying or reversing the current the potential of the coils of the galvanometer is suddenly altered, and unless the moving part can also attain the new potential at the same instant, or be screened from it, electrostatic forces will come into play, causing a false deflection of the instrument.

It is seldom that resistance measurements of precision are required, except between resistances of simple numerical ratio, and such measurements may be divided into the comparison of nominally equal resistances, and of resistances whose ratio is represented by some simple numerical fraction.

Case. I.—Comparison of Nearly Equal Resistances.

This measurement shares with the comparison of nearly equal weights and lengths the position of being capable of being carried out with the highest accuracy attainable in any physical measurement. Differences of 1 part in a million may generally be easily detected between resistances if not less than 1 ohm. It is essential for satisfactory work that they should be capable of carrying a reasonable current.

Method I.—Comparison by Interchanging in Two Adjacent Arms of a Wheatstone Bridge.

The two resistances are made part of a Wheatstone Bridge, A, B, the other two sides, P, Q, being nearly equal resistances, which may or may not be of the same nominal value as A and B. In the latter case A and B should be in series as regards the supply current, as this allows of the largest current to pass through the bridge, and is, therefore, the most sensitive arrangement, and it is also the most convenient. Balance is obtained by shunting one of the resistances with a relatively high resistance. If the bridge is nearly balanced without a shunt, a very high shunting resistance, perhaps a hundred thousand or a million ohms, will be required which is very seldom obtainable. This difficulty can be got over by shunting

one of the other arms, say, Q, by any convenient resistance Q′, whose value is not required to be accurately known, so that the required shunt on P is of convenient dimensions. It has the advantage that the variable resistance is always connected to the same part of the bridge which conduces to the ease and

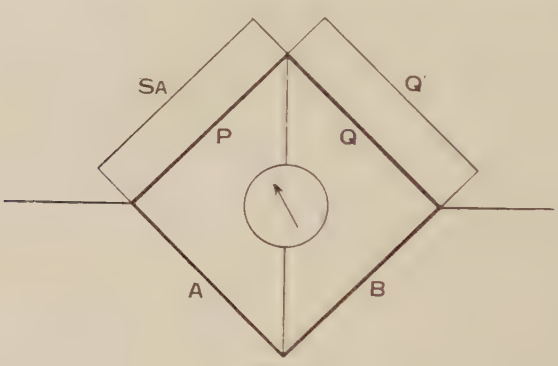

Fig. 1.

quickness of working. Thus, suppose that S_A is the shunting resistance required in parallel with P when A is connected to P, and S_B the shunting resistance when B is next to P.

Then $$\frac{A}{B}=1\div\frac{\left(\frac{1}{P}+\frac{1}{S_A}\right)}{Q} \text{ and also } \frac{A}{B}=\frac{Q}{1\div\left(\frac{1}{P}+\frac{1}{S_B}\right)}.$$

Eliminating Q

$$\frac{A^2}{B^2}=\frac{\frac{1}{P}+\frac{1}{S_B}}{\frac{1}{P}+\frac{1}{S_A}}=\frac{\left(1+\frac{P}{S_B}\right)}{\left(1+\frac{P}{S_A}\right)}.$$

If $\frac{P}{S_B}$ and $\frac{P}{S_A}$ are nearly equal and small compared with unity,

$$\frac{A}{B}\frac{1+\frac{1}{2}\frac{P}{S_B}}{1+\frac{1}{2}\frac{P}{S_A}}=1+\tfrac{1}{2}P\left(\frac{1}{S_B}-\frac{1}{S_A}\right).$$

An example will make the accuracy obtainable more easily appreciated.

Thus P=Q=50 ohms, Q being shunted by a fixed resistance Q′ of about 9,000 ohms.

$A=B=20$ ohms. $S_A=8{,}572$ ohms. $S_B=8{,}493$ ohms.

$$\frac{A}{B}=1+\frac{50}{2}\left(\frac{1}{8{,}493}-\frac{1}{8{,}572}\right)=1+25\times 0{\cdot}000{,}001{,}08=1+0{\cdot}000{,}027.$$

Thus it will be seen that a difference of 80 ohms in the value of the two shunting resistances corresponds to a difference of 27 parts in a million in the ratio $\frac{A}{B}$, and as this difference may often be observable to 0·2 or 0·3 of an ohm, the ratio of A to B may be determined to a few parts in ten millions when A and B

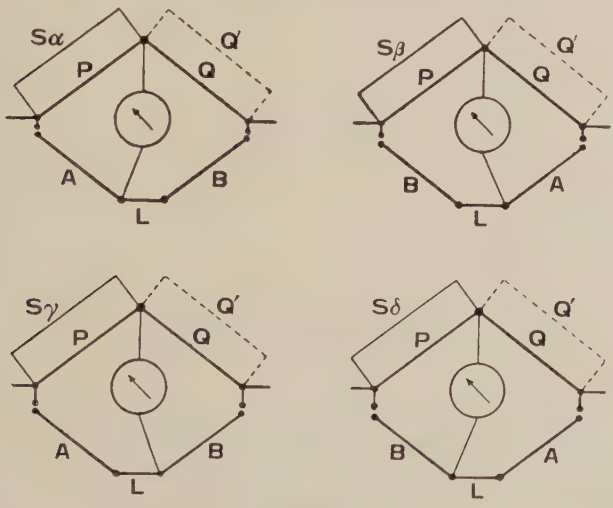

FIG. 2.

are nearly equal. It will be noted that absolute accuracy in P or in S_A and S_B is not required, only that the difference between S_A and S_B should not be seriously in error.

It is important that any leads connecting A and B to P and Q should remain fixed to P and Q, and should not be reversed when A and B are interchanged, otherwise any difference between them will be included in the measured difference between A and B.

In most cases a short connecting piece between A and B is necessary, in which case the method may be extended as follows: The galvanometer is first connected so that the connecting piece of resistance L is first added to B, and the ratio of $\frac{A}{B+L}$ found as above, after which the galvanometer connection is changed so that the ratio $\frac{B}{A+L}$ is obtained, four values of the shunting resistance being observed.

If the value of L is greater than the difference between A and B, then two equations of the form

$$\frac{A}{B+L}=1-x \text{ and}$$

$$\frac{B}{A+L}=1-y \text{ are obtained.}$$

Eliminating L

$$\frac{A}{B}=\frac{1-\frac{y}{2}}{1-y}\times\frac{1-x}{1-\frac{x}{2}},$$

and as x and y are small quantities

$$\frac{A}{B}=\frac{1-\frac{x}{2}}{1-\frac{y}{2}}=1+\tfrac{1}{2}(y-x).$$

To evaluate L, A may be eliminated from the first and last of these equations which leads to

$$\frac{L}{B}=\frac{\frac{1}{2}(y+x)}{1-\frac{1}{2}(1-x)},$$

and as $1-x$ differs little from unity, and $\frac{L}{B}$ is not required to a high accuracy

$$\frac{L}{B}=\tfrac{1}{2}(y+x).$$

Example: A=B=50; P=Q=50. Permanent shunt Q′, of about 9,000 ohms on Q.

The diagrams, Fig. 2, show the connections

$$S_\alpha=8{,}465{\cdot}4,\quad S_\beta=8{,}691{\cdot}8,\quad S_\gamma=8{,}653{\cdot}5,\quad S_\delta=8{,}501{\cdot}5.$$

$$\frac{A}{B+L}=1+\frac{50}{2}\left(\frac{1}{8{,}691{\cdot}8}-\frac{1}{8{,}465{\cdot}4}\right)=1-0{\cdot}000{,}769_2$$

$$\frac{B}{A+L}=1+\frac{50}{2}\left(\frac{1}{8{,}653{\cdot}5}-\frac{1}{8{,}501{\cdot}5}\right)=1-0{\cdot}000{,}516_5$$

$$\frac{A}{B}=1+\tfrac{1}{2}(0{\cdot}000{,}051{,}6_5-0{\cdot}000{,}076{,}9_2)=1-0{\cdot}000{,}012{,}6.$$

which gives the ratio to one part in 10 millions with good accuracy, and

$$\frac{L}{B}=\frac{L}{A}=0{\cdot}000{,}064{,}3.$$

It is easy to estimate what value it is possible to put on the number of significant figures in the decimal part of the ratio A/B, which may be relied on in the final result by looking at the difference between S_α, S_δ, or S_β, S_γ. These are about 36 ohms correct to about 0·1 or 0·2 of an ohm, so that the final error in the decimal part ought to be under 1 per cent., in other words the value $\frac{A}{B}=1-0{\cdot}000{,}012{,}6$ is correct to 1 unit in the last figure. The value $\frac{L}{A}=\frac{L}{B}=0{\cdot}000{,}064{,}3$ must be constant during the experiment, so that being a short length of copper wire its temperature should not alter appreciably. In fact, in all resistance measurements of high precision it is the temperature which puts a limit to the useful accuracy, and constant temperature rooms and oil baths are necessary for the comparison of standard resistances. In the experiments quoted, the observations have to be repeated in the inverse order, and the mean taken to allow for temperature drift. In measurements of this nature each terminal block should have at least two, if not three, terminals in order that the movement of the galvanometer connection from one end of L to the other, and the connection of the high-resistance shunts, &c., may be made without disturbance of any contacts through which the main current passes.

In the example quoted A, B, P and Q were resistances of 50 ohms each wound on open frames. Each resistance was

composed of two circuits of 100 ohms each of No. 24 constantan wire, forming a non-inductive arrangement when the two are used in parallel. They are made so that 100 volts can be put on the frame so as to give 2 amperes, and are regularly used in connection with an accurate electrostatic voltmeter for measuring small alternating currents, a number of frames being put in series or parallel as required. The design is due to Mr. Campbell, and affords the most accurate method of measuring alternating currents of the order of 0·01 ampere to 20 amperes.

The winding is equivalent to two oppositely wound spirals, the current always being led in at one end of the frame and out at the other. If the more usual way of doubling a wire on itself were employed the inductance would be slightly less, but full voltage would exist between the adjacent wires near the ends. The wires carrying oppositely flowing currents are tied together, so as to reduce the inductance as far as possible, and the difference of potential between adjacent wires does not exceed about 3 volts. The frames are about $70 \times 50 \times 1{\cdot}5$ cm. in size, the grooves for the wires in the top and bottom bars being about 1 cm. apart.

It is not necessary to shunt the whole of one of the arms of the bridge by the variable resistance. It is often more convenient and accurate with the resistances available to shunt only a fraction.

The change of resistance with increasing current of one of the (50 ohms) frames capable of carrying 2 amperes, was determined in this manner. The wire for these resistances was specially selected as having a very small temperature coefficient. A bridge was made up of one of these frames as one arm and of nine others (three in series, three in parallel), so as to be equal in resistance in the adjacent arm, it being assumed that the heating in these being so much less would be negligible. The single resistance and the combination of nine were connected in series to a source of supply, so that somewhat over 200 volts could be applied. The other two arms were each of 10 resistances of 1,000 ohms (usable up to 1,000 volts). These resistances, high compared with the galvanometer resistance (10 ohms), rendered the arrangement insensitive at small currents, but the resistance of zero current can be calculated with good accuracy, as will be shown. One-tenth of the resistance Q (1,000 ohms) was shunted by a fixed resistance of about 5,000 ohms, and balance obtained by shunting a similar

section of P by a variable resistance of about the same value. The value of this resistance and the calculation of the relative resistance of P, and, therefore, of A at different currents, is shown in the following table :

Current.	R.	P.	δP.	Current $^{2\cdot6}$.	$\delta P \div C^{2\cdot6}$.
Ampere.					
0·98	5049·0	9834·68	...	0·95	...
1·41	5033·6	9834·26	—0·42	2·44	0·292
1·74	5016·0	9833·78	—0·90	4·22	0·281
1·98	4997·2	9833·26	—1·42	5·93	0·289
2·19	4978·0	9832·78	—1·90	7·68	0·285

The resistance P is the nominal value of the arm P as shunted

$$P=9{,}000+1\div\left(\frac{1}{1{,}000}+\frac{1}{R}\right),$$

δP is the change in P from its value at 0·98 ampere, and is

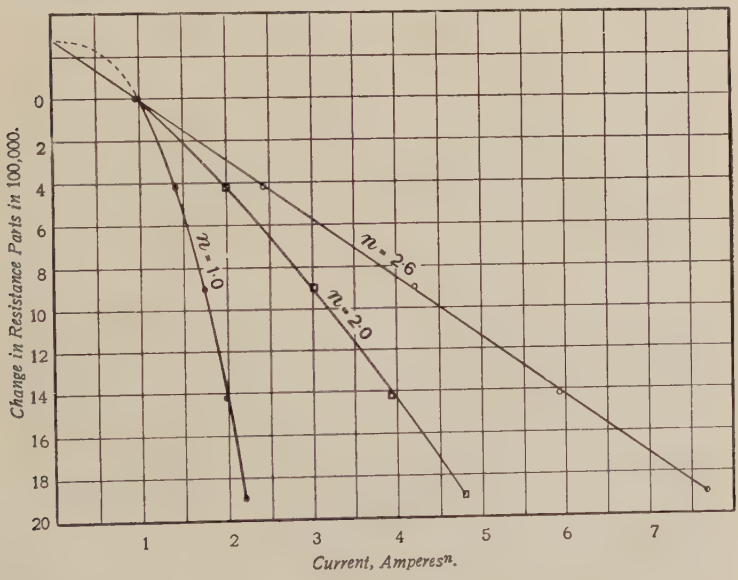

FIG. 3.

practically equivalent to parts in 10,000. From these values (assuming the bridge to be too insensitive for accurate readings below 1 ampere) it is impossible to determine accurately what would be the value of P at zero current by plotting δP against current (n=1) Fig. 3. It is much better to plot δP against the

square of the current which is nearly proportional to the temperature rise ($n=2$). This gives a line with some curvature in it. By trial it will be found that plotting δP against (current)$^{2\cdot6}$ ($n=2\cdot6$), a curve indistinguishable from a straight line results, leading to a value for δP at zero current of $-0\cdot28$. P for 2 amperes is $-1\cdot47$, so that the total diminution in resistance is 1·75 in 9,800, or practically 1·8 parts in 10,000 between the resistance measured at 2 amperes (100 volts) and that at a very small current.

Measurement of Equal Resistances by Substitution.

A very convenient method for the rapid comparison of a number of nearly equal resistances is to make up a Wheatstone bridge with one of them in one of the arms, and to substitute successively the various resistances in turn. The fine adjustment is conveniently made by shunting one of the arms by a high resistance. The other three resistances may be specially wound ones, and be roughly mounted on a board with all joints soldered as far as possible. When adjusting a number of resistances to equality using this method, it is convenient to make up the bridge out of lengths of the same wire, which will avoid temperature changes. The choice of dimensions of the resistances will depend on that of the resistances to be measured and the galvanometer resistance, enabling the conditions of maximum sensitivity to be obtained when desired. For work of the highest precision with standard resistances with heavy copper terminals mercury cups, &c., the method may be inferior in accuracy to the comparison of two resistances by interchanging, but for many practical purposes it is more convenient, as the resistance of the connecting leads is eliminated, and there is more latitude in the choice of the resistances of the other three arms of the bridge.

Case II.—The Comparison of Resistances of Different Magnitudes.

Comparisons of this nature vary much, and only general principles can be suggested. To obtain a ratio of, say, 1 to 10, 10 resistances of the smaller value are put in series, each being capable of direct comparison with the standard. For the methods of building up and verifying resistances of less than 1 000 ohms the B.A. Paper of Mr. Smith should be consulted. Below 1,000 ohms special precautions have to be taken regarding the resistances of the end connections.

Verification of a Potential Divider.

An instructive example in the measurement of comparatively high resistances is that of a potential divider commonly used in conjunction with a potentiometer to measure 100 or 200 volts. A high resistance is connected across the supply voltage to be measured, and exactly 1 per cent. (or 0·5 per cent.) is tapped off and applied to the potentiometer. A voltage of 1·0 or 1·5 per 100 ohms is commonly used, so that for 100 or 150 volts the whole resistance will be 10,000 ohms, with a tapping at 100 ohms for the potentiometer. If standard resistances of 10,000 ohms and 100 ohms are available, the potential divider may be easily verified, provided the resistance of the internal connections can be neglected ; but for precision work this is not sufficient. After being on the circuit for a short time it will be found that the effect of the heating is appreciable. If the whole box changed in resistance homo-

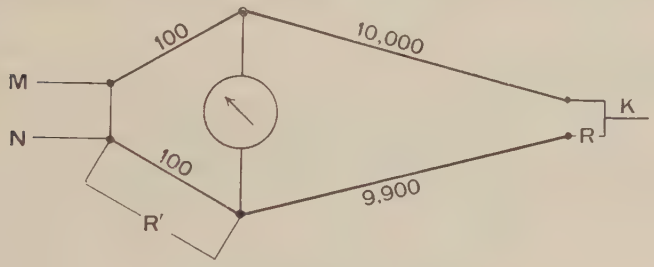

Fig 4.

geneously on this account it would not matter, but the 100-ohm section is often made of thicker wire for ease of adjustment, and even if this is not the case it may remain cooler than the rest, as it is practically always near one end of the resistance box. It is therefore desirable that the ratio of the two parts of the resistance, nominally, 1 : 99 should be determined at the working voltage. To do this we may take advantage of the useful theorem that the value of n nearly equal resistances connected in series is n^2 times their resistance in parallel to a very high order of accuracy. For instance, suppose half of them are as much as 1 in 1,000 high, and the other half 1 in 1,000 low, the n^2 law holds to one part in a million.

Suppose, then, we have 10 resistances of 1,000 ohms, which can be put in series or parallel, and also a resistance of 100 ohms, the 10 of 1,000 ohms in parallel can be compared with the

100-ohm resistance by inversion with very high accuracy. By putting the 1,000-ohm resistances in series, we now have two resistances of a nominal ratio of 100 : 10,000. It is assumed that these resistances are sufficiently liberally designed so as to be unaffected when 100 volts or more is applied to them in series. The volt box ratio to be proved is nominally 100 : 9,900, which may be converted for the purposes of measurement into 100 : 10,000 by two methods. The one is to add a variable resistance of about 100 ohms to the 9,900, making it up to 10,000, and the other is to shunt the 100-ohm section of the volt box with a variable resistance of about 10,000. This is the more delicate method, as 100 ohms change in this resistance is equivalent to 1 ohm in the first method. In the second method 10,000 ohms shunting the 100-ohm section of the volt box reduces the current in it by 1 per cent. The effect of this is neglected. Instead of resistances of 100 and $10\times1,000$ ohms, other convenient resistances of the same ratio can be used. It is possible that the actual resistances of the volt box sections will differ from their nominal value, and correction must be made for this if necessary, when using extraneous resistances in series or parallel with parts of it.

The following is an example from experiments on one of the best makes of commercial potential dividers rated for 150 volts.

Resistances, nominal, of the two sections 100 and 9,900 ohms.

I.—Verification of Nominal Value by Comparison with Known Resistance of 10,000 *ohms.*

Using the nomenclature of Fig. 1.

A = 10,000 known. B = volt box. P = Q = 50. Q′ = 4,000.

$$S_A = 5,711. \quad S_B = 2,987.$$

$$\left(\frac{A}{B}\right)^2 = \frac{2,987+50}{2,987} \times \frac{5,711}{5,711+50} = 1{\cdot}00792.$$

$$\frac{A}{B} = 1{\cdot}003,95.$$

The volt box, as a whole, is therefore low by nearly 40 ohms in 10,000.

II.—Determination of Auxiliary Ratio 100 : 10,000.

A resistance of 10 parts of 1,000 ohms each is arranged to be easily changed from series to parallel. Let F be the ratio of

the 100-ohm resistance to the resistance of the 10 of 1,000 ohms in parallel. From observations similar to those (Fig. 1)

$$F=1+\frac{50}{2}\left(\frac{1}{8{,}649\cdot2}-\frac{1}{8{,}734\cdot1}\right)=1+0\cdot000{,}028.$$

III.—Balancing the Auxiliary Resistance Against Volt Box.

(*a*) *By Addition of a Variable Resistance, R, of about* 100 *ohms.* The testing voltage is supplied at K and at M or N. If supplied at M the resistance of the short wire L is added to the 100 section of the volt box and is appreciable. The mean of the values, found when current is led in at M and N alternately, is taken. Thus, current led in at M, R=95·6 ; in at N, R=96·5 ; mean 96·05. This converted to volt box units ($\times$1·000,395) =96·43. Then

$$\frac{A}{B}=\frac{F}{100}\times\left(1+\frac{R}{B}\right)=\frac{99\cdot969}{9{,}900}.$$

(*b*) *By Shunting the* 100*-ohm Section of the Volt Box.*—Current in at M, R′=10,176 ; current in at N′, R′=10,268 ; mean, 10,222 ; corrected for " volt box unit " 10,263.

$$\frac{A}{B}=\frac{F}{100}\times\frac{10{,}263+100}{10{,}263}=\frac{99\cdot969}{9{,}900}.$$

The two methods agree. These measurements were done with 10 volts.

After the application of 100 volts for half-an-hour the two methods gave $\frac{99\cdot959}{9{,}900}$ and $\frac{99\cdot960}{9{,}900}$, showing a change of 1 part in 10,000 due to the heating, the discrepancy between the two methods being 1 in 100,000.

The change of ratio which is caused by the application of 150 volts for 80 minutes, and the subsequent cooling when 10 volts is substituted is shown in Fig. 5.

The wire is wound round two parallel wooden cylinders separated such a distance apart that about half the total length is stretched in air between the cylinders, and the other half is in close contact with them. The free part very quickly takes up a certain temperature difference above the surrounding air. This accounts for the sudden change during the first minute. The temperature rise of the rest of the wire is

delayed by the heat capacity and conductivity of the wooden cylinders as indicated in the rest of the diagram.

It may be thought that measurements of the ratio of such an apparatus as a potential divider for 100 volts to an accuracy of the order of 1 part in 100,000 is far beyond the accuracy to which an apparatus can be usefully employed. It must, however, be remembered that it is one important link in the maintenance of the voltage of commercial supply networks at their correct value throughout the world.

Though it might not be generally expected, the accurate measurement of voltage is of prime importance in photometry. It is found that a series of photometric comparisons can be made to an accuracy of one part in a thousand. As the candle-power of a lamp varies four or five times as quickly as the

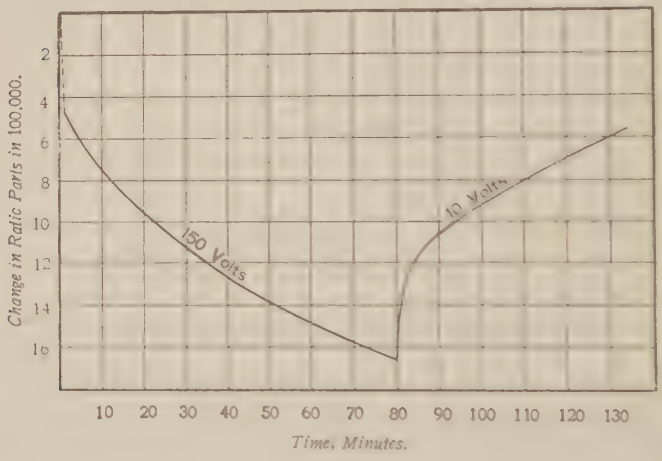

Fig. 5.

voltage, this amount would be accounted for by a change of 1 part in 5,000 in the voltage. It is, therefore, necessary that any errors in the electrical measuring instruments should be known to an accuracy of 1 in 10,000, and measurement of the magnitudes of these errors to the order of 1 in 100,000 is not superfluous.

A simple method of proving the ratio of such potential dividers under working conditions may be of use to makers, as the above results show that there is room for improvement in those at present obtainable.

Use of Continuous and Alternating Currents Superposed.

The author has shown in a Paper* read before the Institution of Electrical Engineers that useful information could be obtained by superposing a continuous potential of a few volts in series with an alternating one of several thousand when testing insulating materials, measuring the amount of continuous current passing, and also the alternating current and power.

The experiments on the volt box just described indicate the importance of measuring their ratios under the working voltage. As continuous voltages of 100 or more are not always so easily obtainable as alternating voltages, and still less is a voltage of 1,000, a similar superposition of a small measuring continuous voltage on a larger heating alternating voltage was tried on the same volt box. The continuous voltage was 13, and the alternating voltage 100. The rate of change of ratio after the application of the alternating voltage was practically the same as if 100 volts continuous were applied. Balance is first obtained with the continuous potential alone, before switching on the alternating potential. The heating effect of the 13 volts is negligible, compared with the 100 volts, the virtual voltage being $\sqrt{100^2+13^2}=100{\cdot}8$. With a moving needle galvanometer it is possible that any alternating current passing through it may affect it, but if a balance with continuous current is first obtained the alternating current will have an insignificant effect. If the time constants of the various parts of the circuit are notably different, which is not likely, as wire of high resistance per unit length is used for such resistances, some disturbance may be caused. This, however, is not likely to be of serious importance, as it is *change* of resistance for some time after application of the full rated voltage that is being looked for, due to rise of temperature, rather than instantaneous effect on switching on the alternating voltage due to differences of time constant.

Kelvin Double Bridge.

For the comparison of unequal resistances, especially when one or both is less than 1 ohm, the Kelvin double bridge cannot be surpassed.

In the comparison of equal resistances of the order of 50 ohms it has been seen that the short piece of wire connecting A and B

* E. H. Rayner, "High Voltage Tests and Energy Losses in Insulating Materials," "Journal of the Inst. of Electrical Engineers," Vol. 49, p. 47, 1912.

affects as much as the last three significant figures attainable in the ratio A : B. If it were practicable to connect the galvanometer exactly to the resistance centre of this wire, allowing for the uncertain contact resistances at the ends, then it would be unnecessary to double the number of observations which are required to eliminate its resistance, Fig. 6. The elegant method of surmounting this difficulty adopted in the Kelvin bridge is to connect the galvanometer, not to the middle point of the wire, but to the middle point of a resistance in parallel with it which is high in comparison with that of the joining wire, Fig. 7. The ends of this resistance should not be joined to the terminals through which the main current passes, but to a separate pair (the potential points of the resistance when these are provided). The resistance may be from 2 to 200 ohms, according to circumstances. In the general case where the two main resistances to be compared are not equal, this connecting resistance must be divided in the ratio of the main resistances.

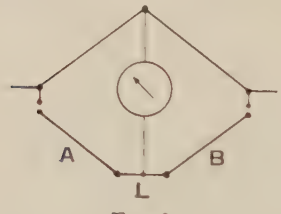

Fig. 6.

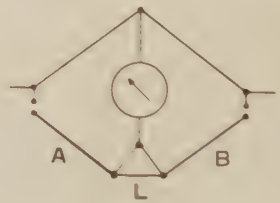

Fig. 7.

The resistances P, Q, are connected to the other potential points of the two resistances.

In commercial forms of the apparatus P is commonly made equal to R and Q equal to S, Fig. 8. Taking A to be a standard resistance, P and R may be of some simple value of about 20 to 100 ohms, and Q and S can be varied together from 0·1 to 1,000 ohms, and the ratio of B to A is read off at once to four figures. The sensitivity of the arrangement increases if low resistances, rather than high ones, are used for P, Q, R, S, but the resistance of the connecting leads makes 10 or 20 ohms the lower limit.

If the resistance of L is small compared with A and B, the values of R and S need not be so accurate as those of P and Q, and it becomes practicable to make R less than P, and S less than Q, with an increase in sensitivity without appreciable loss of accuracy. When measuring small resistances of 0·001

ohm or less, when L may be as large or larger than A or B, accuracy in R and S is nearly as important as in P and Q. The correctness of R/S may be verified by breaking the main circuit at L, when it will be seen that the system ought still to remain balanced. Practically the whole voltage of the source of supply will now come on to R and S in series, and care must be taken that these are not damaged if small resistances are used.

The equation representing the general condition that there be no current through the galvanometer is that

$$B=A\cdot\frac{Q}{P}+\frac{RL}{R+S+L}\left(\frac{Q}{P}-\frac{S}{R}\right).$$

As in the case of resistances of an ohm or more, precision

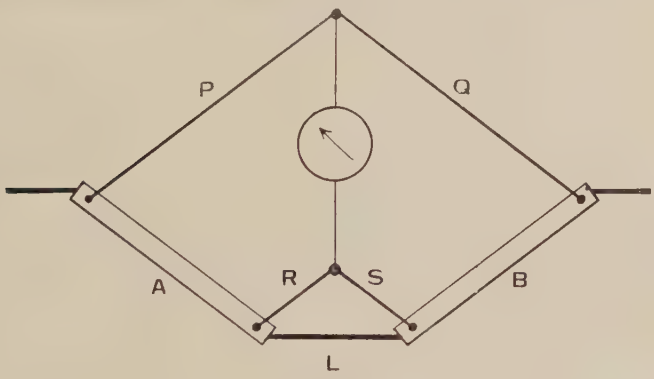

FIG. 8.

measurements are only required as a rule between resistances having a simple nominal ratio which enables the apparatus to be considerably simplified.

Special Form of Kelvin Bridge.

The necessity for apparatus of this nature for verifying the resistances used in precision alternating electrical measurements has led to the construction of a special Kelvin bridge for the purpose. The resistances to be verified vary from a few ohms to 0·001 ohm, and all have a simple numerical value. They are used in the current circuit when making alternating power and current measurements, and give a 2-volt drop of potential at their rated current.

The apparatus consists of two rows of resistances of 20 ohms each, there being 25 in each row. One row forms the part

P, Q, and the other R. S. as many of the resistances being used as required and connections are taken off for the galvanometer circuit.

As the resistances of the leads connecting the apparatus to the resistances to be compared cannot be made negligibly small, compared with 20 ohms, the end coils of the two rows have been adjusted to be 1 part in 1,000 low (0·05 ohm), and the leads used are adjusted to be 0·05 ohm.

These end coils must, therefore, always be used, and the bridge is adjusted to the correct simple ratio by short-circuiting

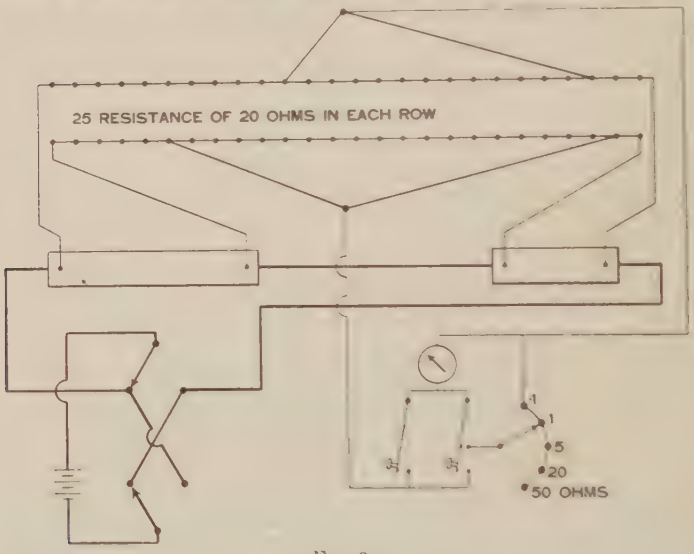

Fig. 9.

as many of the other coils as may be necessary. For convenience the two rows of 25 resistances are put in concentric circles, and copper straps are used for short-circuiting the necessary coils, these straps being also used for the galvanometer connections, which are arranged concentrically at the centre of the circle. The arrangement has the great merit that all the resistances have the same value, and so can be adjusted to equality with high accuracy. The absolute accuracy is quite immaterial. Balance is obtained by shunting A or B, whichever is relatively the higher.

The apparatus is capable of comparing any resistances of the ratio $\frac{m}{n}$, where m and n are any integers whose sum does

not exceed 25. Though it is possible to measure ratios of 24 : 1, lower ones such as 10 : 1 and 5 : 1 can be somewhat more accurately measured, as then at least two coils can be used in each of P and Q, and the small uncertainties of the resistances of the temporary connections, &c., become of less importance.

In the diagram the bridge is shown arranged for measuring a ratio of 5 : 1, P consisting of 10 coils, Q of 2, R of 5 and S of 1. It has been found satisfactory to mount the terminals directly on the wooden base which is soaked in hot wax. The holes through the wood are tapped, and the terminals screwed up tight, and locked by brass nuts in the usual way. This effectually prevents the terminals working loose, which is inevitable in wood if they pass through clearance holes.

The arrangement shown for the galvanometer keys has been found very satisfactory. When one key is depressed the galvanometer is first of all shunted by a resistance which can be varied from 0·1 to 50 ohms, and further depression makes a second contact connecting the galvanometer to the circuit. When approximate balance has been obtained the other key is depressed instead, which connects the galvanometer to the circuit without any shunt. It has been found much more satisfactory than the usual arrangement of a key with a resistance in circuit when half depressed which is cut out on applying more pressure. A mercury cup reversing key for the main current is provided.

In order to compare a resistance of, say, 0·001 ohm with one of 1 ohm it is necessary to do so in a series of steps, using intermediate resistances of, say, 0·1 and 0·01 ohm with a ratio of 10 : 1 in the bridge resistances every time. Any inequalities in the bridge resistances will cause a cumulative error in the final result, so that they should be very accurately adjusted to equality. These differences have been kept down to 2 or 3 parts in 100,000, so that in a "step-down" of 3 steps the accumulated error does not exceed 1 in 10,000.

Resistances of Incommensurable Ratios.

The accurate measurement of resistances of this nature is very seldom required. There is one case, however, in connection with the use of standard cells which is of practical importance. In apparatus of a potentiometer nature it is usual to adjust the current very exactly to some simple value, such as 0·01 ampere, and the easiest way to do so is to pass it through a resistance of 100 times the nominal value of the

voltage of a Weston cell ($100 \times 1{\cdot}0183$ ohms at 20°C.). If the potential divider has a number of equal resistances in series, each designed to have 10 volts on its terminals, it is a simple matter to connect a number of these, mostly in parallel, such that their resistance is exactly that to which the cell resistance has to be adjusted, which can therefore be done with great accuracy. Sixteen such resistances arranged in a certain manner will give a resistance equivalent to the voltage 1·0184, which is the voltage of Weston cell at 17°C. Other combinations may be made which will give slightly different values corresponding to the voltage of the cell at other temperatures. The subject has been dealt with in a previous publication.*

General Remarks.

The following points should receive attention in such measurements as have been described :—

1. The current should be as large as practicable, having regard to the change of resistance with temperature.

2. It should be measured by an ammeter.

3. In order to enable large currents to be used resistances should be liberally designed.

4. For ratio arms and other resistances frequently used it is of great advantage to wind them on open frames. The terminal blocks should have three terminals each. Two or three resistances of this nature of each of such values as 10, 100 and 1,000 ohms each will be very useful. They may be wound with two circuits so as to be very non-inductive.

5. Ten equal resistances which can be quickly changed from series to parallel are very useful. In order to avoid uncertainties due to contacts and differences of current paths in the two arrangements, 1,000 ohms each is desirable. These may be wound on micanite, which allows of better cooling than several layers on a coil. If heavy copper ends and mercury cups are used, 100-ohm units may be used instead of 1,000 ohms.

6. In Wheatstone bridge arrangements final balance is best obtained by a shunt of relatively high resistance in parallel with some part of the circuit. The higher the relative resistance the more delicate is the adjustment, and the less the importance of the accuracy of the shunting resistance. The system may very

* Paterson, Rayner and Kinnes, "The Use of the Electrostatic Method for the Measurement of Power," "Journal of the Inst. of Electrical Engineers," Part 221, Vol. 51, p. 315, 1913.

often be put intentionally out of balance to enable a suitable shunting resistance to be employed.

7. It is generally worth while, as far as possible, to design apparatus to be "self-checking," *i.e.*, in such a manner that the ratios of the various parts can be easily determined without the use of an external resistance whose value is required to be accurately known. Absolute accuracy of adjustment of resistances, though convenient, is not essential in most laboratory apparatus. It is only the relative proportions of the parts that are of importance in such instruments as potential dividers, potentiometers and Kelvin double bridges. It is only when resistances are used for the measurement of current that absolute accuracy becomes of importance.

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ABSTRACT.

The Paper describes methods by which the comparison of resistances can be made to an accuracy of 1 in 10,000 or higher by using simple apparatus usually available in electrical laboratories, or which can be easily constructed with little skilled assistance.

The comparison of nominally equal resistances of 1 ohm and upwards by the usual method of shunting one side of a nearly balanced quadrilateral by a high resistance is mentioned, and variations on this when only part of one resistance is shunted are often useful. The great advantages of having resistances capable of carrying comparatively large currents is illustrated, especially for measuring changes of resistance of commercial apparatus under working conditions. The determination of errors in a volt box for use with a potentiometer is described at some length. This is of especial importance in precision photometry.

If a sufficient continuous-current voltage is not available for testing such apparatus as high-potential dividers, it is shown that using sufficient continuous current to secure sensitivity the heating may be supplied by superposed alternating current.

Resistances in common use are very generally of simple numerical value, and a Kelvin bridge specially designed for the comparison of such resistances is described. It consists essentially of two rows of 25 resistances of 20 ohms each.

DISCUSSION.

Colonel SQUIER thought the Paper was very useful and likely to be generally helpful.

Mr. F. E. SMITH made several suggestions which, he thought, would improve the Paper. First, the conventional lettering of the arms of the bridge should be adopted, and an equation showing the sensitiveness of an ordinary bridge should be inserted. The Kelvin bridge arrangement could not be brought forward too prominently. If the equation representing the balancing condition was inserted it would be clear to all what errors might be introduced if the resistance of this lead was too high. With regard to the interchange method of comparing nominally equal resistances it was not the best or simplest for coils with potential leads. Suppose P, Q, R, and S (Fig. 10) each have a nominal resistance of 1 ohm, but that P is provided

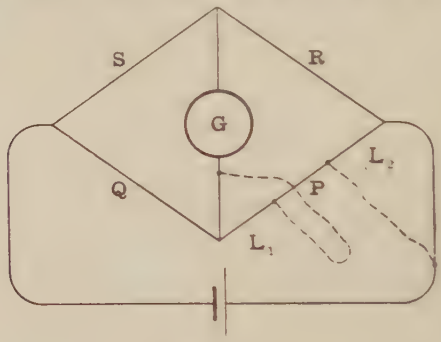

FIG. 10.

with potential leads, and it is the value of P between these which is desired. Let L_1 and L_2 be the resistance of the leads of P. Then, when the connections are as in full line we have

$$P + L_1 + L_2 = \frac{QR}{S'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

S′ being the shunted value of S necessary to obtain a balance.

By means of a switch the connections are now changed to those shown by the dotted lines, when

$$P = \frac{(Q + L_1)(R + L_2)}{S''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

S″ being the new shunted value of S.

Adding (1) and (2) and neglecting very small quantities we have

$$2P = QR\left(\frac{1}{S'} + \frac{1}{S''}\right).$$

The sensitiveness of this method is identical with the method of interchange, and it is much more convenient for coils with potential leads. The thermal E.M.F.s in a bridge, especially with coils containing Ni are often troublesome, and in the method of interchange the E.M.F. through the galvanometer is often reversed on interchanging. This should be remedied by reversing the galvanometer connection, or it may at times be remedied by reversing the position of a coil. Regarding heating effects, about three years ago, in collaboration with Dr. Glazebrook and Mr. Boustield, I published some results on the increase of resistance with increase of current, but in no instance was any

difficulty found in obtaining the resistance for an infinite small current by plotting the change of resistance against C^2. Of course, all the curves were not straight lines. With resistance alloys such as constantan and manganin we do not expect them to be. The rise in temperature was, however, in all cases strictly proportional to C^2, and the temperature resistance curve could be plotted from the observation. Does the author think the constantan he worked with behaved differently, and if there was any real advantage in plotting change of R against $C^{2\cdot6}$ instead of C^2. Mr. Rayner's Paper will undoubtedly be of value to those who have to measure resistances with fair precision. There is, however, a bridge which for accuracy, ease of working and range (if suitably designed) is difficult to surpass. I refer to certain types of thermometer bridge. In potential, then, no matter what the resistance may be, between certain values, it has to be measured with very great accuracy. The bridge is readily calibrated, and requires no external apparatus for building up values. It is readily adapted to measure any resistance from about 0·1 ohm to 1,000 ohms, and the coils may have potential leads. I fear that because some bridges have been designed for precision work they are regarded as unsuitable for other work of a lower degree of accuracy. This fortunately is not true.

Mr. Duddell said that with regard to the building up of resistances he had in his laboratory a large number of 1,000 ω coils which were capable of carrying 0·1 ampere. By arranging these units in series and parallel combinations he could readily obtain any ratio he wished, and the building up of a potentiometer with these coils was very easy and convenient. An advantage in the use of high resistance coils such as 1,000 ω for work of this kind was the relative unimportance of the connection errors, which were very small if the resistances had substantial plug contacts.

Mr. A. Campbell said that he had introduced the system of non-inductive winding on chanelled frames about 16 years ago, and it had been publicly in use in the National Physical Laboratory for many years. It appeared to have been patented recently by a foreign firm. The lessening of the inductance is obtained by winding a left and right-handed coil in the same set of channels and connecting them in parallel, tying the contiguous pairs of wires closely together. The capacity is well distributed if many channels are used. In a simpler method, which he has also used for many years, a single wire is twisted into a large number of loops, in each of which the wires are very close together, and each channel carries a single loop.

Mr. C. C. Paterson said that it was inevitable when working with ordinary resistances on a high voltage that a good deal of heat should be dissipated. In potential dividers one makes the coils all of the same wire, measures the ratios at low currents and relies on these remaining the same at high currents. This is not the case, however, the ratio altering by an appreciable amount depending on the precise position of the portion from which the voltage is picked off—*e.g.*, at the end or the middle of the coils—on account of unequal temperature distribution. He instanced a case in which errors of two or three parts in 10,000 were introduced from this cause, and said that the method of superposing an alternating voltage on a small direct-current voltage was very useful for determining the ratios under the working conditions of heat dissipation.

Mr. D. Owen welcomed a Paper on this subject. Emanating from the National Physical Laboratory, it would be regarded as embodying the soundest methods for use under the simple conditions which alone are generally attainable. The claim (in the first test quoted) of an accuracy of one part in 10 millions under simple conditions, when neither temperature nor the value of the testing current was specified, was misleading, and apt to propagate the mischievous illusion that *sensibility* of the bridge and

accuracy of the measurement were one and the same thing. The proof put forward of the formula for the method of interchange was, unfortunately, not rigorous, as the resistance of the leads at the outer ends of P and Q was not taken into account. The extension of the method which the author advocated, requiring four balances, appeared unnecessarily cumbrous: by connecting the galvanometer to a terminal fixed midway along the lead L the number of balances was reduced to two without loss of accuracy. In the test of effect of current strength on resistance of a coil the neglect of the heating effect in the adjacent arm was unjustifiable, as it amounted to 10 per cent. of the variation to be determined.

Dr. C. V. Drysdale (communicated): The two great principles upon which all standard bridges are based are those of Carey-Foster and of Kelvin, and in 1907 I combined both these principles in a bridge which is simple in construction, capable of working to the highest possible accuracy with resistances of any value, very rapid in use, and which gives the difference between the standard and test resistances directly in millionths of their value. This is when the two resistances to be compared are practically equal, and it is preferable and nearly always possible in a good laboratory at the present time to have a standard of approximately the same value as the resistance to be determined. In order, however, to arrange for stepping up or down, a self-contained ratio bobbin was devised embodying the principle laid down by Lord Rayleigh, having five coils of relative values 1, 3, 3, 3, 1. By suitable and convenient combination of these coils, which is now effected by solid bars and mercury contacts, not only can most of the uneven ratios required be obtained, but the arrangement is self checking, and it forms an accurate series of standards built up from a single unit. It is true that when uneven ratios are employed, shunting has to be substituted for the slide-wire, but this is only necessary when comparing or adjusting a series of standards of different values, and it can be done with perfect facility on the same bridge. Apart from the convenience of adjustment, Mr. Rayner's bridge appears to suffer from the defects that contact errors in connecting his ratios are not eliminated, and that it is not self-checking. Except where the resistances have very appreciable and different time constants, also, it is decidedly bad practice to use a galvanometer key at all. Alterations of sensitiveness should be secured by varying the main current. Mr. Rayner correctly states that great advantage as regards contact errors can be secured without great loss of sensitiveness by keeping the resistance of the ratio coils moderately high, instead of attempting to fulfil the mathematical conditions for maximum sensitiveness. In practice it is possible to work to an accuracy of one or two parts in 10,000,000, when the ratio coils are never reduced below 10 or even 100 ohms.

Mr. Rayner replied as follows: As to lettering the different arms of the bridge P, Q, R, S, which Mr. Smith says is the conventional manner, neither the Americans nor the Germans seem to have adopted a conventional notation, and I have been troubled in following the discussion of various methods by the difficulty in keeping in mind the significance of the main and secondary resistances when P, Q, R and S are used. As to the change in resistance varying as the 2·6th power of the current, Mr. Smith considers the effect is most likely due to the non linearity of the temperature coefficient rather than divergence from Newton's law of cooling, which his experiments on resistance thermometers seem to show hold with great exactness. Mr. Owen considers the neglect of an equal amount of heat generated in the nine other frames may account for it. I do not think it would make much difference, as the change of resistance at such a small temperature rise is very small. The point is not of great importance, as in practice a slight draught of air will make a considerable difference. The experiment is only quoted as an example of a method, and to give an idea what the effect may be in the case of resistances of a certain design intended for a much higher voltage and current than is usually employed. Mr. Smith has drawn my attention to the

fact that the statement in the description of Fig. 1 that for greatest sensitiveness resistances of similar values should always be put in series is incorrect. In practice, however, as resistances of different denominations are usually designed to withstand the same watts and not to carry the same current, the arrangement in the Paper is nearly always preferable. He also states that on reversing an equal armed bridge the potential of the galvanometer does not vary. This is the case if the reversing switch in the battery circuit connects directly to the bridge. If there is a regulating resistance for adjusting the main current added between the reversing switch and the bridge, then the potential of the galvanometer circuit is changed. For this reason an adjusting resistance should be put on the supply side of the reversing switch. I agree with Mr. Duddell as to the size of unit to be used when resistances are to be put in series or parallel. They should not be less than 1,000 ohms unless errors are carefully guarded against. Regarding the errors due to change of ratio of potential dividers mentioned by Mr. Paterson, the application of the method of superposing an alternating on a continuous potential for the accurate measurement of the effect was largely his suggestion. I agree with Mr. Owen that a sensitiveness to such a figure as, say, 1 in 10,000,000, does not necessarily imply similar accuracy of measurement, and the proof of the full use of the sensitivity lies in reproducing the numerical value on repeating the measurement. All the examples given have been reproducible in this sense. This does not mean that identically the same shunting resistances have been required on repeating a series of measurements, as very small temperature changes in a few minutes will affect the result; but it does mean that if, for instance, on repeating a given experiment, a different shunting resistance is required in the one position, an exactly equivalent change is found in the other position, showing that the measurement takes full advantage of the sensitivity available. There is all the difference between the measurement of small differences of temperature to a small fraction of a degree, such as has been suggested in the Paper, and the measurement of temperature generally to the corresponding fraction of a degree. Platinum silver, perhaps, one of the most stable materials for use as a permanent standard, is quite unsuitable as a working resistance by reason of its comparatively large temperature coefficients. Manganin and constantan can usually be obtained with a temperature coefficient of one-tenth the value. The suggestion of connecting the galvanometer to the middle of L in Fig. 2 is impracticable by reason of the resistance of the end contacts. Dr. Drysdale in his communication suggests the use of a combination of Carey-Foster and Kelvin bridges, using a bridge wire for the fine adjustment. It could hardly be termed a simple apparatus which could be made in any laboratory workshop, such as has been the aim of the Paper to describe. With bridge wire methods the thermoelectric force generated at the point of contact of the galvanometer circuit and the bridge wire is liable to cause trouble. It will vary in intensity when the contact is moved, and if contact with the wire is maintained to mask the effect some wear takes place. If the contact is broken before movement takes place the galvanometer moves from its false zero and must come to rest before the main current can be switched on. These and other difficulties have caused Mr. Smith to discard such methods in favour of shunting methods, in which the galvanometer circuit is always closed. The precision measurement of certain resistances of high and low values has been forced on us by the continually increasing demands for accuracy in technical electrical measurements, and the methods which have been described in the Paper have been adopted for some of these largely because they could be satisfactorily used to the accuracy required with the simple apparatus available. For a higher accuracy than that comtemplated in this Paper reference should be made to the papers named in the bibliography. Dr. Drysdale's series of articles in "The Electrician" give a most valuable account and criticism of standard resistances.

XXX. *On Electrically-maintained Vibrations.* *By* S. BUTTERWORTH, *M.Sc., Lecturer in Physics, School of Technology, Manchester.*

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1. In a Paper published some time ago by the author.* it was shown that when a vibrating system of one degree of freedom was set in motion by the action of a current, then, if the forces acting are proportional to the current, the system behaves as a parallel combination of a capacity, a conductance, and an inductance.

In some experiments arising from the theory, use was made of a wire carrying alternating currents, and stretched between the poles of an electromagnet. This system does not conform to the type discussed in that it enjoys an infinite number of degrees of freedom. It is proposed in the present Paper to extend the theory so as to include the possibility of a large number of degrees of freedom.

The only limitations imposed on the motion of the systems considered will be those usually assumed in the theory of small motions—viz., that the displacements from a position of stable equilibrium are so small that their squares may be neglected, and that the frictional forces are proportional to the velocities.

Two classes of displacing forces will be considered :—

(A) Those that arise when a coil carrying currents is in the neighbourhood of permanently magnetised bodies. The mechanical force at any point is then proportional to the current in the coil.

(B) Those due to the action of a charged condenser on electrified bodies. The mechanical force is then proportional to the E.M.F. across the plates of the condenser.

2. If the system possesses only one degree of freedom, its equation of motion is

$$(\alpha D^2+\beta D+\gamma)y=Y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $D=\frac{d}{dt}$, α, β, γ are the constants of inertia, damping and restoration respectively, y is the displacement, and Y the displacing force.

For systems of class A,

$$Y=AI, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* Butterworth, "Proc." Phys. Soc., XXVI., p. 264, 1914.

where I is the current through the coil. Hence, in this case, the power to be supplied to maintain the motion is AIDy, or the E.M.F. due to the motion is

$$e=\mathrm{AD}y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating y from (3) by means of (1) and (2),

$$e=\mu\mathrm{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $$\mu=\mathrm{A}^2/(\alpha\mathrm{D}+\beta+\gamma/\mathrm{D}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence, if r and l are the resistance and inductance of the coil and E its terminal E.M.F.,

$$\mathrm{E}=(\mu+r+l\mathrm{D})\mathrm{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the motion of the system be prevented, and we place in series with the coil a parallel combination of a capacity (C_a), a conductance (S_a), and an inductance (L_a), the new terminal E.M.F. is

$$\mathrm{E}'=(\mu'+r+l\mathrm{D})\mathrm{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $$\mu'=1/(\mathrm{C}_a\mathrm{D}+\mathrm{S}_a+1/\mathrm{L}_a\mathrm{D}), \quad . \quad . \quad . \quad . \quad (8)$$

(6) and (7) are identical if $\mu'=\mu$, that is

$$\mathrm{C}_a=\alpha/\mathrm{A}^2,\ \mathrm{S}_a=\beta/\mathrm{A}^2,\ \mathrm{L}_a=\mathrm{A}^2/\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For systems of Class B,

$$\mathrm{Y}=\mathrm{BE}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where E is the E.M.F. across the plates of the condenser. Hence, in this case the power to be supplied to maintain the motion is BEDy, or a current

$$i=\mathrm{BD}y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

must be supplied in addition to that maintaining the charges on the condenser plates. Eliminating y from (11) by means of (1) and (10),

$$i=\nu\mathrm{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $$\nu=\mathrm{B}^2/(\alpha\mathrm{D}+\beta+\gamma/\mathrm{D}). \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hence, if s and c are the conductance and capacity of the condenser, and I the total current,

$$\mathrm{I}=(\nu+s+c\mathrm{D})\mathrm{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If the motion of the system be prevented, and we place in parallel with the condenser a series combination of an induc-

tance (L_b), a resistance (R_b) and a capacity (C_b), the new total current is

$$I'=(\nu'+s+cD)E, \quad \ldots\ldots (15)$$

where $$\nu'=1/(L_bD+R_b+1/C_bD), \quad \ldots\ldots (16)$$

(14) and (15) are identical if $\nu'=\nu$, that is,

$$L_b=a/B^2,\ R_b=\beta/B^2,\ C_b=B^2/\gamma. \quad \ldots\ldots (17)$$

3. We now develop the corresponding theory when the system enjoys m degrees of freedom.

Let the generalised co-ordinates be $y_1, y_2 \ldots y_m$, and the generalised velocities be $\dot{y}_1, \dot{y}_2 \ldots \dot{y}_m$. Then, for small motions about a position of stable equilibrium, the kinetic energy (T), the dissipation function (F), and the potential energy (V) are given by

$$\left.\begin{aligned} 2T&=\Sigma a_{rr}\dot{y}_r^2+2\Sigma a_{rs}\dot{y}_r\dot{y}_s \\ 2F&=\Sigma \beta_{rr}\dot{y}_r^2+2\Sigma \beta_{rs}\dot{y}_r\dot{y}_s \\ 2V&=\Sigma \gamma_{rr}y_r^2+2\Sigma \gamma_{rs}y_ry_s \end{aligned}\right\} \quad \ldots\ldots (18)$$

Using Lagrange's method we obtain the m equations of motion, of which the rth is

$$\frac{d}{dt}\left(\frac{dT}{d\dot{y}_r}\right)-\frac{dT}{dy_r}+\frac{dF}{d\dot{y}_r}+\frac{dV}{dy_r}=Y_r, \quad \ldots\ldots (19)$$

where Y_r is the generalised force corresponding to the co-ordinate y_r.

Let the co-ordinates be so chosen that F and V reduce to sums of squares, so that

$$\beta_{rs}=\gamma_{rs}=0,$$

and substitute the values of T, F, and V in equations (19). The rth equation now becomes

$$Y_r=(a_{rr}D^2+\beta_{rr}D+\gamma_{rr})y_r+\Sigma a_{rs}D^2y_s, \quad \ldots\ldots (20)$$

in which $D=\dfrac{d}{dt}$.

For systems of class A the displacing force at any point is proportional to the current, so that

$$Y_r=A_rI. \quad \ldots\ldots (21)$$

The power to maintain the motion is then given by ΣY_rDy_r, or the E.M.F. due to the motion is

$$e=\Sigma A_rDy_r=\Sigma e_r \text{ (say)}, \quad \ldots\ldots (22)$$

where $$e_r=A_rDy_r. \quad \ldots\ldots (23)$$

Eliminating y_r by means of (23) in (20), and using (21),

$$A_r I=(\alpha_{rr}D+\beta_{rr}+\gamma_{rr}/D)\frac{e_r}{A_r}+\Sigma\alpha_{rs}De_s/A_s. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If α_{rs} were zero the equivalent electrical system would be obtained by combining in series m parallel combinations of capacity, conductance and inductance, the values of which for the rth combination would be

$$C_{rr}=\alpha_{rr}/A_r^2,\ S_{rr}=\beta_{rr}/A_r^2,\ L_{rr}=A_{rr}^2/\gamma_{rr}, \quad . \quad . \quad . \quad (25)$$

for this combination would satisfy the current-E.M.F. equations (24) and the relation $e=\Sigma e_r$.

When α_{rs} is not zero, the series arrangement of condensers must be replaced by a condenser network. The equivalent electrical system is then built up as follows: Connect $m+1$ points $P_0, P_1..P_m$ by condensers such that every point is connected to every other point once and once only. $m(m+1)/2$ condensers will be necessary. Let K_{rs} be the capacity of the condenser connecting P_r to P_s. Join successive points (P_{r-1}, P_r) by parallel combinations of conductance and inductance (S_{rr}, L_{rr}). Let a current I enter the system at P_0 and let it leave at P_m.

Then, if $i_1,..i_{r+1}—i_r,..—i_m$ are the currents entering the condenser network at $P_0,..P_r,..P_m$ respectively, i'_r is the current through the conductance S_{rr}, i''_r is the current through the inductance L_{rr}, we have

$$I=i_1+i'_1+i''_1=i_2+i'_2+i''_2=....=i_m+i'_m+i''_m.... \quad . \quad (26)$$

Also, if v_r is the potential of P_r

$$\left.\begin{aligned} i_1&=K_{01}D(v_0—v_1)+K_{02}D(v_0—v_2)+....\\ i_2—i_1&=K_{10}D(v_1—v_0)+K_{12}D(v_1—v_2)+....\\ &.....\end{aligned}\right\}$$

in which $K_{rs}=K_{sr}$.

Replacing the actual potentials by P.D.s between successive points (that is, putting $v_{r-1}—v_r=e_r$) and solving for $i_1, i_2..$,

$$\left.\begin{aligned} i_1&=C_{11}De_1+C_{12}De_2+....\\ i_2&=C_{21}De_1+C_{22}De_2+....\\ &.....\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

in which $C_{rs}=C_{sr}$.

Also, $i'_r=S_{rr}e_r$, $i''_r=e_r/L_{rr}D$. . . . . (28)

Adding equations (28) to the rth equation of (27) we have from (26)

$$I=(C_{rr}D+S_{rr}+1/L_{rr}D)e_r+\Sigma C_{rs}De_s. \quad . \quad . \quad . \quad . \quad (29)$$

Again, if e is the total P.D. between P_0 and P_m,

$$e=\Sigma e_r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

A comparison of equations (29) and (30) with (24) and (22) shows that the electrical system is equivalent to the mechanical system provided that (25) holds, and

$$C_{rs}=a_{rs}/A_rA_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

For systems of class B, the displacing force is proportional to the E.M.F. (E) across the plates of a condenser, so that in (20)

$$Y_r=B_rE. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The power to maintain the motion is, as before, ΣY_rDy_r, so that by (32) the current * to be supplied to the condenser in order to maintain the motion is

$$i=\Sigma B_rDy_r=\Sigma i_r \text{ (say)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where

$$i_r=B_rDy_r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Eliminating y_r in equation (20) by means of (34) and using (32),

$$B_rE=(\alpha_{rr}D+\beta_{rr}+\gamma_{rr}/D)i_r/B_r=\Sigma a_{rs}Di_s/B_s. \quad . \quad . \quad . \quad . \quad (35)$$

The equivalent electrical system follows immediately from (33) and (35). It is simply m series combinations of inductance, resistance and capacity arranged in parallel, and possessing mutual inductance between the various coils. If L_{rr}, R_{rr}, C_{rr} are the values of the inductance, resistance and capacity for the rth combination, and L_{rs} is the mutual induction between L_{rr} and L_{ss}, the conditions for equivalence are

$$\left.\begin{aligned} L_{rr}&=a_{rr}/B_r^2, \quad R_{rr}=\beta_{rr}/B_r^2 \\ C_{rr}&=B_r^2/\gamma_{rr}, \quad L_{rs}=a_{rs}/B_rB_s \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

4. A stretched wire in a transverse magnetic field affords the simplest illustration of a system of class A enjoying an infinite number of degrees of freedom.

Instead of following the general method of section 3, the normal equations will be derived from the partial differential equation of motion.

* This current exists in addition to the charging and leakage currents.

Let the wire be fixed at two points at distance $2l$. Take the centre of the wire as origin and the undisplaced position of the wire as the axis of x. If the intensity of the magnetic field at x is H and a current I flows through the wire, the equation of motion is

$$m\frac{\partial^2 y}{\partial t^2}+\rho\frac{\partial y}{\partial t}=T_1\frac{\partial^2 y}{\partial x^2}+HI, \quad . \quad . \quad . \quad . \quad . \quad (37)$$

in which T_1 is the tension, m the linear density, and ρ the damping per unit length. The end conditions are

$$y=0 \text{ when } x=-l, \text{ and } x=+l \quad . \quad . \quad . \quad (38)$$

Whatever the form of the wire when displaced we may expand y in a Fourier series which holds from $x=-l$ to $x=+l$. Thus, let

$$y=\sum_{r=1}^{r=\infty}(y_r \sin a_r x+z_r \cos b_r x), \quad . \quad . \quad . \quad (39)$$

a_r, b_r are determined by the conditions (38) giving

$$a_r=r\pi/l,\ b_r=(2r-1)\pi/2l, \quad . \quad . \quad . \quad . \quad . \quad (40)$$

in which r is an integer.
H may be expressed in a similar series, viz.,

$$H=\sum_{r=1}^{r=\infty}(H_r \sin a_r x+K_r \cos b_r x). \quad . \quad . \quad . \quad (41)$$

Substituting (39) and (41) in the equation of motion (37)

$$\sum_{r=1}^{r=\infty}\{(m\ddot{y}_r+\rho\dot{y}_r+T_1 a_r^2 y_r-H_r I)\sin a_r x\cos b_r x\}$$
$$+(m\ddot{z}_r+\rho\dot{z}_r+T_1 b_r^2 z_r-K_r I)=0, \quad . \quad . \quad (42)$$

whence the normal equations of motion are

$$\left.\begin{aligned}(mD^2+\rho D+T_1 a_r^2)y_r&=H_r I\\(mD^2+\rho D+T_1 b_r^2)z_r&=K_r I\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (43)$$

Before we can apply the results of section 3, it may be necessary to multiply these equations by a constant factor. To find this factor it is only necessary to form the expression for the kinetic energy.

We have
$$2T=m\int_{-l}^{l}\dot{y}^2 dx$$
$$=ml\sum_{r=1}^{r=\infty}(\dot{y}_r^2+\dot{z}_r^2)$$

by (39) and (40).

Referring to (18), $a_{rr}=ml$, so that (43) is identical with (20) if we multiply the former by l. We therefore have

$$\left.\begin{aligned} &a_{rr}=ml,\ \beta_{rr}=\rho l \\ &\gamma_{rr}=T_1 a_r^2 l \text{ when } A_r=H_r l \\ &\gamma_{rr}=T_1 b_r^2 l \text{ when } A_r=K_r l \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$a_{rs}=0.$$

The values of H_r and K_r are determined by the form of H when expressed as a function of x: if H is an even function of x, H_r vanishes, if an odd function K_r vanishes.

For example, suppose H to be zero when $x>a$ and $x<-a$, and H to be constant when $a>x>-a$. Then by the usual method,

$$H_r=0,\ K_r=4H \sin (2r-1)\pi a/2l/(2r-1)\pi \quad . \quad . \quad (45)$$

5. Returning to the case of one degree of freedom, a system of class A behaves as a parallel combination of capacity, conductance and inductance, in series with an inductive coil, while a system of class B behaves as a series combination of inductance, resistance and capacity in parallel with a condenser.

The electrical equivalent of system A is unrealisable owing to the absence of resistance in the inductance, but the electrical equivalent of system B can be realised if the damping is sufficiently large. Moreover, a system of class B may be made to balance a system of class A on a Wheatstone bridge by placing the two systems in the conjugate arms (Q, R) and pure resistances (S, P) in the other arms.

The condition of balance is then by (6) and (14)

$$\frac{\mu+r+l\mathrm{D}}{\nu+s+c\mathrm{D}}=\mathrm{SP},$$

so that by (8) and (16) the balance will be independent of the form of the current if

$$\mathrm{L}_b/\mathrm{C}_a=\mathrm{R}_b/\mathrm{S}_a=\mathrm{L}_a/\mathrm{C}_b=r/s=l/c=\mathrm{SP}.$$

Hence for large dampings the constants of system A can be found by balancing with the electrical equivalent of system B.

This method may be extended to the general case. The resistance operator (μ) of system A is obtained by eliminating the m E.M.F.s ($e_1, e_2 \ldots e_m$) from the $m+1$ equations (29) and (30). This gives

$$-\Delta_1 \mathrm{I}=\Delta'_1 e,$$

where

$$\Delta_1 = \begin{vmatrix} 0 & 1 & 1 & 1 & .. \\ 1 & k_{11} & k_{12} & k_{13} & .. \\ 1 & k_{21} & k_{22} & k_{23} & .. \\ 1 & k_{31} & k_{32} & k_{33} & .. \\ .. & .. & .. & .. & .. \end{vmatrix} \quad \Delta'_1 = \begin{vmatrix} k_{11} & k_{12} & k_{13} & .. \\ k_{21} & k_{22} & k_{23} & .. \\ k_{31} & k_{32} & k_{33} & .. \\ .. & .. & .. & .. \end{vmatrix}$$

in which

$$k_{rr} = C_{rr}D + S_{rr} + 1/L_{rr}D$$

$$k_{rs} = C_{rs}D = k_{sr}.$$

Hence

$$r = -\Delta_1/\Delta'_1.$$

Similarly for system B the resistance operator is

$$1/v = -\Delta'_2/\Delta_2,$$

where

$$\Delta_2 = \begin{vmatrix} 0 & 1 & 1 & 1 & .. \\ 1 & l_{11} & l_{12} & l_{13} & .. \\ 1 & l_{21} & l_{22} & l_{23} & .. \\ 1 & l_{31} & l_{32} & l_{33} & .. \\ .. & .. & .. & .. & .. \end{vmatrix} \quad \Delta'_2 = \begin{vmatrix} l_{11} & l_{12} & l_{13} & .. \\ l_{21} & l_{22} & l_{23} & .. \\ l_{31} & l_{32} & l_{33} & .. \\ .. & .. & .. & .. \end{vmatrix}$$

in which

$$l_{rr} = L'_{rr}D + R'_{rr} + 1/C'_{rr}D$$

$$l_{rs} = L'_{rs}D = l_{sr},$$

the accents being introduced to distinguish system B from system A.

The condition for a complete balance when the two systems are in conjugate arms of a Wheatstone bridge is as before that μ/ν shall be independent of D. This will be the case if

$$\frac{L'_{rr}}{C_{rr}} = \frac{R'_{rr}}{S_{rr}} = \frac{L_{rr}}{C'_{rr}} = \frac{L'_{rs}}{C_{rs}} = SP.$$

The constants of a system of class B can be found by balancing with its equivalent electrical system in the adjacent arm of a Wheatstone bridge.

6. The balances discussed in the previous section are mainly of theoretical interest only. If we use alternating currents of frequency $p/2\pi$, then the measurements may be considerably simplified.

For a system of class A and of one degree of freedom, if the natural frequency of the vibrating system is $p_0/2\pi$ so that

$$p_0^2 = 1/L_a C_a = \gamma/a, \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

then when there is no damping and $p<p_0$ the system behaves as an inductance

$$L_0=L_a/(1-n^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

in which $n=p/p_0$,

and as a capacity

$$C_0=C_a(1-1/n^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

if $p>p_0$.

These results follow from (8) with $S_a=0$. If S_a is not zero, then we must suppose S_a to be placed in parallel with L_0 or C_0. In this case the moving system behaves as an inductance

$$L'_0=L_0/(1+p^2L_0^2S_a^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (47a)$$

in series, with a resistance

$$R'_0=p^2L_0^2S_a/(1+p^2L_0^2S_a^2) \quad . \quad . \quad . \quad . \quad (47b)$$

when $p<p_0$

and as a capacity

$$C'_0=C_0+S_a^2/p^2C_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (48a)$$

in series with a resistance

$$R'_0=S_a/(S_a^2+p^2C_0^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (48b)$$

when $p<p_0$.

For a system of class B and of one degree of freedom the system behaves as a capacity $\{C_a/(1-n^2)\}$ in series with a resistance (R_b) if $p<p_0$, and as an inductance $\{L_a(1-1/n^2)\}$ in series with the same resistance if $p>p_0$.

If the system possesses many degrees of freedom, the mode of procedure is to arrange the degrees of freedom in the order of their natural frequencies. If of class A, those below the frequency of the current behave as capacities and those above behave as inductances. By combining these inductances and capacities in series, we obtain the resultant effect as a single inductance or capacity.

A similar process will hold for class B, using parallel combinations for the various degrees of freedom.

The procedure thus outlined only holds when $a_{rs}=0$.

As might be inferred from general considerations the preponderant effect is due to those degrees of freedom whose frequencies lie in the immediate neighbourhood of the source frequency.

The case of a wire in a magnetic field will now be considered in detail.

7. It will be supposed that the damping of the wire is negligible except when the source frequency is in the immediate neighbourhood of one of the natural frequencies of the wire.

If the magnetic field is symmetrical about the centre of the wire then the frequencies of the possible normal modes of vibration are odd multiples of the fundamental frequency which by (40) and (44) is given by

$$p_1^2=\frac{\pi^2}{4l^2}\frac{T_1}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

If the source frequency is in the immediate neighbourhood of the fundamental of the wire, all the harmonics of the wire behave as inductances of which the rth has the value

$$L_{rr}/\{1-n^2/(2r-1)^2\},$$

in which $n=p/p_1$.

Further, when the field is uniform throughout the length of the wire, we have by (45), (44) and (40)

$$L_{11}=\frac{64}{\pi^4}\frac{H^2l^3}{T_1},\ L_{rr}=L_{11}/(2r-1)^4 \quad . \quad . \quad . \quad . \quad (50)$$

Hence, the apparent inductance due to the wire harmonics is

$$l=L_{11}\sum_{r=2}^{r=\infty}\frac{1}{(2r-1)^4\{1-n^2/(r-1)^2\}}.$$

By expanding each term of this series in ascending powers of n, it may be written

$$l=L_{11}(a_4+a_6n^2+a_8n^4+\ldots.), \quad . \quad . \quad . \quad . \quad (51)$$

where $a_r=3^{-r}+5^{-r}+y^{-r}+\ldots..$

The values of a_r are known (Dale's Tables, p. 92). Thus :—

$$a_4=0{\cdot}01468,\ a_6=0{\cdot}00145,\ a_8=0{\cdot}00015,\ a_{10}=0{\cdot}00002.$$

The series (51) will give 1 per cent. accuracy for l so long as the source frequency is less than 1·4 times the fundamental 10 per cent. up to twice the fundamental. In the latter case l is less than $0{\cdot}02L_{11}$.

If the field, instead of being uniform, tapers off towards the ends of the wire, the effect of the harmonics is less than that estimated above. Hence, for frequencies up to $2p_1$ we may treat the wire as a system of one degree of freedom with a small extra inductance in series with it.

8. An estimate of the inductance L_{11} for the wire may be obtained as follows: Let the wire be of material of density δ and specific resistance ρ. Then, if the cross-section is a, the resistance R is $2\rho l/a$, and the mass per unit length m is $a\times\delta$. Hence, by (49) and (50),

$$\frac{L_{11}p_1^2}{R}=\frac{8H^2}{\pi^2\rho\delta}, \quad \ldots\ldots\ldots \quad (52)$$

a quantity independent of the length and cross-section of the wire.

Thus, for phosphor-bronze, $\rho\delta=10^5$ c.g.s., so that if a wire of this material is tuned to a frequency of 100 per second and is placed in a uniform field of strength 10,000 c.g.s.,

$$L_{11}\doteqdot 2 \text{ millihenries per ohm},$$

while for other frequencies of tuning and other field strengths L_{11} is proportional to H^2/p_1^2.

9. The notation employed in the preceding sections has been used for convenience in developing the theory. For experimental purposes with systems of class A and of one degree of freedom it is preferable to revert to the notation and nomenclature of an earlier Paper.* A vibrating system of this type behaves as a parallel combination of capacity, conductance and inductance. We will denote the elements of this combination by C_v, S_v, and L_v, and refer to them as the *vibration capacity*, *vibration conductance*, and *vibration inductance* respectively. Since their values are independent of the frequency of the current employed, they will be spoken of generally as the *vibration constants*. It has been shown in the Paper quoted how these constants can be measured on an alternating-current bridge, in which a balance is obtained which is independent of the form of the current.

The method is, however, difficult to carry out, and for most purposes it is preferable to make use of the results of section 6 and to treat the moving system as an inductance and series resistance (equations $4y_a$ and $4y_b$) when the source frequency is less than that of the vibrating system: as a capacity and series resistance when the source frequency is greater than that of the vibrating system. Except for large dampings the apparent inductance or capacity is given to a sufficient degree of accuracy by (47) or (48).

* Butterworth, "Proc." Phys. Soc., XXVI., p. 264, 1914.

Hence, if the source frequency is f and the natural frequency is f_0, the apparent inductance (L_0) is given by

$$L_0 = L_v/(1-n^2), \quad \ldots\ldots (53)$$

in which $n = f/f_0$, when $f < f_0$, and the apparent capacity (C_0) by

$$C_0 = C_v(1-1/n^2), \quad \ldots\ldots (54)$$

when $f > f_0$.

A consideration of the magnitudes of L_0 and C_0 for a given system shows that it is in general simpler to measure L_0, although by using some form of capacity bridge, say, the

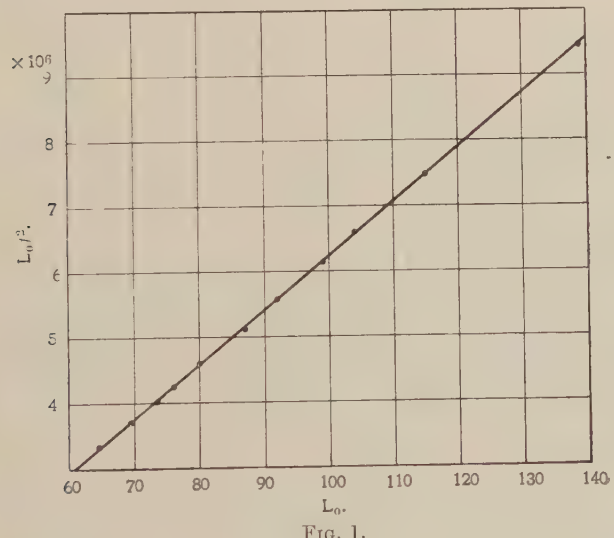

FIG. 1.

modified Carey-Foster bridge, the apparent capacity C_0 could no doubt be measured.

Table I. shows how the apparent inductance of a vibrating wire was found to vary with the frequency of the source. In order to check whether the experimental values satisfy (53) the values of L_0 are plotted against L_0/f^2 in Fig. 1. The result is a straight line, which is in accordance with (53). From the constants of this line,

$$f_0 = 287, \quad L_v = 24{\cdot}3 \text{ microhenries.}$$

Using these values and recalculating L_0 from (53) the values

given in the last column in the table are derived. It is seen that they agree with the observed values to within 2 per cent.

TABLE I.—*Variation of Apparent Inductance* L_0 *of Vibrating Wire with Source Frequency* (f).

f ~ per sec.	L_0 microhenries.	$L_0 f^2$.	L_0 (calculated).
228	64·5	$3{\cdot}35\times10^6$	65·9
231	69·5	3·71	69·0
234	73·5	4·03	72·5
237	76·0	4·27	76·4
240	80·0	4·61	80·7
243	87·0	5·13	85·5
246	92·0	5·57	91·4
249	99·0	6·14	98·0
252	104·0	6·60	106·0
255	115·0	7·48	114·8
261	139·0	9·47	139·8

By varying the magnetising current exciting the electromagnet used to produce the field the inductances tabulated in Table II. were obtained. In these observations the source frequency was maintained at 240~ per second and the tension increased, the effect of increase of tension being to reduce the inductance, but at the same time to reduce the effect of fluctuations in source frequency. Since the apparent inductance (L_0) varies as the square of the field H, it follows that if we plot $L_0^{\frac{1}{2}}$ against the magnetising current we obtain the form of the hysteresis loop for the electromagnet. This is done in Fig. II.

TABLE II.—*Variation of* L_0 *with Magnetising Current* (I) *of Electromagnet.*

I Amperes.	L_0 (microhenries).				
	Current direct.		Current reversed.		Current direct
	Rising.	Falling.	Rising.	Falling.	Rising.
0·00	...	1·0	...	1·0	...
0·23	22·4	23·4	16·5	23·0	16·5
0·30	25·6	28·8	22·3	28·4	22·3
0·40	31·8	34·8	30·2	34·4	30·4
0·50	37·4	38·8	35·2	38·8	34·8
0·60	39·6	41·0	38·2	41·6	38·4
0·70	41·6	43·2	41·2	42·8	41·0
0·80	42·6	43·8	...	44·2	42·2
0·90	43·8	44·8	44·2	45·4	43·8
1·00	44·8	46·0	44·8	46·6	45·0
1·40	49·2	...	49·6	...	48·2

10. The results just quoted were obtained for a loop of phosphor-bronze wire of 10 cm. vibrating length and kept

in tension by means of a spring. The arrangement was placed in the air-gap of an electromagnet, the width of the air-gap being 2 mm. and the pole faces 6×4 cm. The measurements were made on a modified Anderson's bridge,* using as detector a Duddell vibration galvanometer. The source was a small alternator, whose fundamental frequency could be varied from 60 to 100$\sim$ per second. The wave-form of the source indicated that it possessed a pronounced third harmonic. As the detector could only be tuned to a minimum

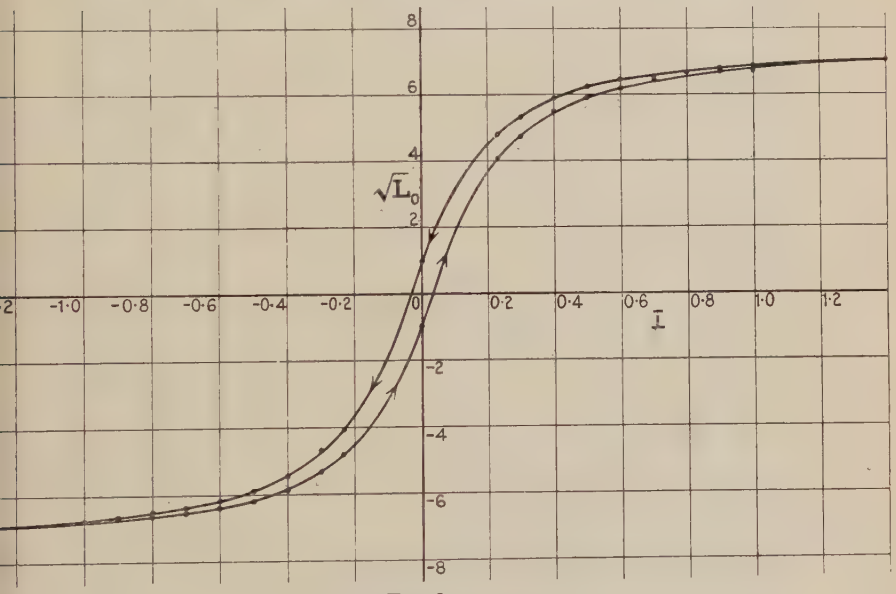

FIG. 2.

frequency of 150$\sim$ per second, the measurments relate not to the fundamental of the source, but to the third harmonic. The fundamental frequency of the source was generally measured by a reed-frequency meter, but as this instrument was not always available use was sometimes made of a Campbell frequency bridge.† The results were concordant. In order to reduce the influence of the fundamental a wave-form sifter,‡ suitable for low frequencies, was employed.

* Butterworth, "Proc." Phys. Soc., XXIV., p. 210, 1912. "Electrician," Vol. LXIX., p. 777, 1912.

† Campbell, "Phil. Mag.," Vol. XV., p. 155, 1908.

‡ Campbell, "Proc." Phys. Soc., XXIV., p. 107, 1912.

11. An interesting application of the theory for systems of class B is supplied by the phenomena of absorption and residual charge in condensers. It has been pointed out * that these phenomena can be imitated electrically by connecting in parallel with a perfect condenser, another condenser (or set of condensers) with a series resistance. Now, the present theory shows that an electrically charged system when set in motion by a varying charge on a condenser, behaves as a set of series combinations of inductance, resistance and capacity arranged in parallel across the original condenser. If the free periods of the dynamical system are high the capacity effect is preponderant, so that the dynamical system would cause the original condenser to exhibit the phenomena of absorption and residual charge. The theory, however, is not sufficiently comprehensive to explain by atomic or other motions the whole of the phenomena exhibited by dielectrics, as only small motions about a position of stable equilibrium are assumed in the dynamical equations.

12. *Summary.*

1. A vibrating system of one degree of freedom when set in motion by the interaction of a current on a magnetic field is shown to behave as a parallel combination of capacity, conductance and inductance.

2. When set in motion by the interaction of charged bodies on an electrostatic field it behaves as a series combination of inductance, resistance and capacity.

3. The corresponding electrical equivalents for systems of many degrees of freedom are obtained.

4. Methods of balancing the systems on alternating-current bridges are discussed, and it is shown how the constants of the systems may be determined by such bridge measurements.

5. Experiments illustrating the theory in the case of a vibrating wire are quoted.

6. It is shown that a moving system of charged bodies is capable of explaining the phenomena of absorption and residual charge in condensers.

In conclusion, I wish to thank Prof. W. W. Haldane Gee for the interest he has taken in the work, and Mr. W. W. Stainer, B.Sc., for assistance in the experimental work.

* Fleming, "Proc." Inst.E.E., Vol. XLIX., p. 323, 1912.

XXXI.—*Numerical Relationships between Electronic and Atomic Constants.* *By* H. Stanley Allen, *M.A., D.Sc., Senior Lecturer in Physics at University of London, King's College.*

Received February 25, 1915.

One of the chief aims of the physicist is to reduce the number of quantities in terms of which natural phenomena can be expressed, that is to increase the number of derived units at the expense of the so-called fundamental units. For example, the four units length, mass, time and charge of electricity may be reduced to three by assuming that mass is wholly electromagnetic in origin. The quantum-theory has introduced a new quantity in Planck's constant h. At the Birmingham meeting of the British Association Jeans pointed out that hc, where c is the velocity of light, has the same physical dimensions as the square of an electric charge. "In point of fact, $hc/2\pi$, if not exactly equal, is almost equal to $(4\pi e)^2$, *i.e.*, to the square of the strength of a tube of force binding two electrons. This suggests that the atomicity of h may be associated with the atomicity of e."*

This equivalence between h and e is strongly supported in an interesting Paper by Lewis and Adams†, who conclude that the quantum is merely the square of the fundamental unit of electricity, e, with a simple numerical coefficient depending on the units chosen. They believe that, ultimately, "all universal constants will prove to be pure numbers, involving only integral numbers and π," and decide, without hesitation, "to take the electron charge as the first of the two fundamental constants needed for the final determination of the ultimate units."

Their procedure may be briefly summarised as follows:—

The gas law may be written $PV=nRT$, where n is the number of mols. If m is the total number of molecules and N is the number in one mol, $m=nN$, and the gas law becomes

$$PV=m(R/N)T=mkT,$$

where k is a universal constant, which always occurs to the same

* Jeans, "Report on Radiation and the Quantum-Theory," Physical Society of London, p. 80, 1914.

† Lewis and Adams, "Phys. Rev.," Vol. III., p. 92, 1914.

power as T. The value of k, the gas constant reckoned for one molecule, is Re/F, where F is the Faraday equivalent in the same units as e. This gives $k=1{\cdot}372\times10^{-16}$.

The fundamental assumption made by Lewis and Adams is that the constant a of Stefan's law, $E=aVT^4$, can be expressed in the form

$$a=k^4/(4\pi e)^6.$$

The occurrence of $4\pi e$ here and elsewhere in their results need cause no surprise, as it would be easy to invent a system of units in which $4\pi e$ is the unit of charge.

This assumption gives for the numerical value of a, $7{\cdot}60\times10^{-15}$, from which we may obtain at once as the value of the radiation constant,

$$\sigma=\frac{ac}{4}=5{\cdot}70\times10^{-5},$$

which is in excellent agreement with recent experimental determinations.

Again, by integrating the Planck equation we obtain

$$a=\frac{8\pi^5k^4}{15c^3h^3}$$

and identifying this value of a with that previously assumed we get

$$15c^3h^3=8\pi^5(4\pi e)^6,$$

or

$$ch=\sqrt[3]{\frac{8\pi^5}{15}}(4\pi e)^2.$$

This is the equation giving h in terms of the electron charge. Taking Millikan's value for $e=4{\cdot}774\times10^{-10}$ E.S.U., this gives for h, $6{\cdot}558\times10^{-27}$, which again is in good agreement with the accepted value.

Two Numerical Constants.

The relation between h and e obtained by Lewis and Adams may be written in the form

$$\frac{4\pi^2e^4}{h^2c^2}=\frac{(15/\pi^2)^{\frac{1}{3}}}{(4\pi)^4}=p, \text{ say.}$$

Here p is a pure number whose value is $5{\cdot}30096\times10^{-5}$.

We shall find it convenient later to put $p=q^2$ where

$$\frac{2\pi e^2}{hc}=\frac{(15/\pi^2)^{\frac{1}{6}}}{(4\pi)^2}=q.$$

and q is $7{\cdot}28077\times10^{-3}$.

Electronic Constants.

Now it is a curious fact that several of the fundamental constants of Physics, expressed in the usual electrostatic units, contain simple multiples of the number p.

The charge on the electron, according to the careful determination of Millikan* is

$$\mathbf{e}=4{\cdot}774\pm0{\cdot}009\times10^{-10}.$$

If we put $\mathbf{e}=9p\times10^{-6}$ we obtain

$$\mathbf{e}=4{\cdot}771\times10^{-10},$$

which agrees with Millikan's value within 0·1 per cent.

Bucherer† has reviewed the more recent determinations of $\mathbf{e/m}$.

Classen, 1908	$1{\cdot}776\times10^{7}$ E.M.U.
Bucherer, 1908	$1{\cdot}763\times10^{7}$
Wolz, 1909	$1{\cdot}767\times10^{7}$
Malassez, 1911.....	$1{\cdot}769\times10^{7}$
Bestelmeyer, 1911.....	$1{\cdot}766\times10^{7}$

If we put $\frac{\mathbf{e}}{\mathbf{m}}=p\times10^{22}$ we obtain

$$\frac{\mathbf{e}}{\mathbf{m}}=5{\cdot}30096\times10^{17}\text{ E.S.U.}$$

or

$$=1{\cdot}767\times10^{7}\text{ E.M.U.},$$

in close agreement with the most recent determinations. The error is not likely to exceed 0·1 per cent.

The value of $\mathbf{m}$, the mass of the electron, is determined from these two results. Consequently within very close limits we obtain

$$\mathbf{m}=(9p\times10^{-6})\div(p\times10^{22})$$
$$=9\times10^{-28}\text{ gm.}$$

In view of the fact that the fundamental units, the centimetre, the gram and the second are in one sense arbitrary, it is difficult to regard the occurrence of exact powers of 10 in these expressions as other than accidental, just as we regard the approximate expression‡ for the velocity of light, viz., 3×10^{10}, as a coincidence. The repeated coincidences do, however, suggest a question as to whether there may not be some accidental connection between the units of length, mass and

* Millikan, "Phys. Rev.," Vol. II., p. 143, 1913.

† Bucherer, "Ann. d. Physik," Vol. XXXVII., p. 597, 1912.

‡ The accurate value is smaller than the conventional one by about 1 part in 3,000.

time involving the number 10. Such a connection may arise from the way in which the units were originally defined. The metre was intended to be one ten-millionth of the quadrant of the earth from the equator to the pole. The second is defined with reference to the period of axial rotation of the earth. If we express the velocity, v, of a point on the equator in terms of the "natural unit of velocity" c, we find $\frac{v}{c}=\frac{1}{2^3\times 3^4\times 10^3}$. If it were possible to express such a velocity in terms of the mass and dimensions of the earth, a relation between the fundamental units could be established. But as no satisfactory theory determining the rate of rotation of a planet has ever been given, there appears no hope of further progress in this direction.

The only alternative is to consider the possibility of a connection arising from the definition of the gram. The kilogram was intended to have the same mass as that of a cubic decimetre of water at the temperature of its maximum density. Now, if N be Avogadro's constant,* one gram of water contains $N/18$ atoms of oxygen and $N/9$ atoms of hydrogen. The atomic volume of hydroxylic oxygen, according to Le Bas, is exactly twice the atomic volume of hydrogen. Consequently if we knew the exact volume to be assigned to the atom of hydrogen (assumed spherical), it should be possible to determine the volume occupied by the given number of atoms when the density is a maximum. Unfortunately we are at present unable to specify the atomic volume of hydrogen, that is its sphere of action, in terms of primary constants. It is, however, suggestive to find a cube root occurring in the numerical constant q. This might be introduced in expressing the diameter of the sphere of action in terms of the volume. We conclude that although it is not possible to determine the form of the connection between the centimetre and the gram from our present knowledge, such a connection must exist, and there is no reason why it should not involve the number 10.

Derived Quantities.

If we accept the expressions for e, m and c in the forms

$$e=9p\times 10^{-6}, \qquad m=9\times 10^{-28}, \qquad c=3\times 10^{10},$$

as being at least arithmetically convenient, it is clear that we

* It is shown later in the Paper that N is approximately $\frac{p}{9}\times 10^{23}$.

can express all derived units depending upon them in terms of simple integers (2, 3 or 4), powers of 10, p and π. As some of these derived quantities contain $\sqrt{p}$ it will be convenient to put $p=q^2$ and use q in our expressions.

The following results may serve as illustrations :—

$$e=(3q\times 10^{-3})^2,$$

$$e/m=(q\times 10^{11})^2,$$

$$m=(3\times 10^{-14})^2,$$

$$h/2\pi=\text{the unit of angular momentum}=(3q)^3\times 10^{-22},$$

$$he/4\pi mc^2=5 \text{ magnetons}=27q^5/2c^2=\frac{3}{2}\left(\frac{q}{10^4}\right)^5,$$

$$a=\text{radius of electron}=\frac{2}{3}\left(\frac{q}{10}\right)^4,$$

$$A_H=\text{radius of H atom on Bohr's theory}=\left(\frac{q}{100}\right)^2,$$

$$\nu_0=\text{fundamental frequency}=\frac{2\pi^2me^4}{h^3}=\frac{3\times 10^{14}}{4\pi q},$$

$$N_0=\text{Rydberg's constant}=\frac{\nu_0}{c}=\frac{10^4}{4\pi q}.$$

The last result is interesting as affording a means of testing the degree of accuracy to be obtained by these approximations. Substituting the value of q we find $N_0=109{,}300$ instead of the value 109,679 found by Curtis, or 109,724, Bohr's value for a massive nucleus. The calculated value is smaller than the observed by less than 0·4 per cent.

Atomic Constants.

If we turn from quantities connected with the electron to quantities connected with the atom, we find results of a similar character, but the numerical agreement is not nearly so good. The fundamental quantity in connection with the atom is Avogadro's constant, N, the number of molecules in one mol. The estimates which have been given of this number vary widely, but there can be no doubt that the most accurate value is that derived from electrical measurements. We take Millikan's value of $e=4{\cdot}774\times 10^{-10}$. "Since the value of the Faraday constant has now been fixed virtually by international agreement (atomic weight of silver 107·88, E.C.E. of silver 0·01118) at 9,650 electromagnetic units, and this is the number

N of molecules in a gram molecule times the elementary electrical charge, we have

$$N \times 4{\cdot}774 \times 10^{-10} = 9{,}650 \times 2{\cdot}9990 \times 10^{10},$$
$$N = 6{\cdot}062 \times 10^{23}.''$$

From the value of N we can deduce at once the mass of the atom of hydrogen, m_H, for $m_H = \frac{1}{N}$. This gives $m_H = 1{\cdot}649 \times 10^{-24}$ gm. The value of m/m_H the "atomic mass of an electron," may be determined without assuming a knowledge of the value of e. For we have, taking Bucherer's final value for e/m,

$$\frac{e}{m} = 5{\cdot}307 \times 10^{17},$$

and

$$\frac{e}{m_H} = 9{,}650 \times 2{\cdot}999 \times 10^{10}.$$

Hence

$$\frac{m}{m_H} = 5{\cdot}453 \times 10^{-4}.$$

This suggests taking $\frac{m}{m_H} = 10p$, in which case

$$\frac{m}{m_H} = 5{\cdot}30096 \times 10^{-4},$$

and there is a difference of about 3 per cent. between the observed and the assumed value. On this assumption the mass of the hydrogen atom would be equivalent to 1,886 electrons, we should have

$$m_H = m/10p = 9 \times 10^{-29}/p,$$

and

$$N = \frac{p}{9} \times 10^{29} = 5{\cdot}89 \times 10^{23},$$

which is about 3 per cent. smaller than Millikan's value.*

It would, of course, be possible to express other quantities depending on the mass of the hydrogen atom in the same fashion, but in view of the considerable error involved, no

* A further comparison with experimental determinations is rendered possible by assuming that the magnetic moment of an electron revolving with the unit of angular momentum is equivalent to 5 magnetons. For the revolving electron the magnetic moment $= 2\pi q^2 2c = 3{\cdot}0689 \times 10^{-31}$ E.S.U. $= 92{\cdot}066 \times 10^{-22}$ E.M.U. If we divide Weiss's value for the magnetic moment of the atom gram, 1123·5, by Millikan's value for Avogadro's constant we obtain as the magnetic moment of the magneton $18{\cdot}54 \times 10^{-22}$ E.M.U., and for 5 magnetons $92{\cdot}7 \times 10^{-22}$ E.M.U. The calculated and the experimental values differ by less than 1 per cent.

useful purpose would be served. The results suggest that the relation between the mass of the hydrogen atom and that of the electron is of a more complicated character than the relations between the electron constants.

ABSTRACT.

It has been pointed out by Jeans that hc, where h is Planck's constant and c is the velocity of light, has the same physical dimensions as the square of an electric charge. Lewis and Adams have suggested a relation between these quantities of the form

$$ch = \sqrt[3]{\frac{\pi^5}{15}}(4\pi e)^2.$$

This may be written

$$\frac{2\pi e^2}{hc} = \frac{(15/\pi^2)^{1/3}}{(4\pi)^2} = q.$$

where q is $7{\cdot}28077 \times 10^{-3}$. The square of this numerical constant is $p = 5{\cdot}30096 \times 10^{-5}$. The charge e on an electron in E.S.U. is found to be, within 0·1 per cent., $9p \times 10^{-6}$. The ratio e/m of the charge to the mass is found to be $p \times 10^{22}$, with the same order of accuracy. The occurrence of powers of 10 in these expressions may be accidental or may depend on the way in which the units of length, mass and time were originally defined. Derived quantities, depending on e, m and c, can be expressed in terms of simple integers (2, 3 or 4), powers of 10, p and π. The "atomic mass of an electron" is approximately $10p$.

DISCUSSION.

Prof. O. W. Richardson said it was curious that the value of e and e/m should happen to be simple multiples of the quantity p. These two quantities were the key to the situation, as the other relations followed more or less directly from them.

Mr. S. D. Chalmers said it was possible almost with any set of quantities to obtain striking numerical relationships. It was only when a physical significance was sought for that it could be seen whether they were anything but coincidental.

Dr. W. Wilson said that if the units of mass, length, temperature difference, &c., were so chosen as to make the universal constants such as Planck's h, the gravitation constant, &c., each equal to unity, the unit of mass which had to be adopted was p gram.

Prof. Zeleny emphasised the importance of searching for the physical significance attached to these relationships.

XXXII. *On a Method of Calculating the Absorption Coefficients of Different Substances for Homogeneous X-radiation.* By H. Moore, *A.R.C.S., B.Sc., Assistant Lecturer in Physics, University of London, King's College.*

Received April 30, 1915.

It has been shown as a result of the work of various observers,* that the ionisation produced in a gas when a beam of X-rays passes through it is due to electrons liberated in the gas by the X-rays. If we take two gases at the same pressure, and subject them to identical beams of X-rays, the numbers of electrons liberated in the gases per unit length of the X-ray beam are not identical, the quantity of electronic radiation liberated in a gas depending on the chemical nature of the gas. This electronic radiation is an atomic phenomenon: it depends solely on the nature and numbers of the atoms constituting the gas, being entirely independent of the way in which these atoms are combined.†

In a recent Paper ("Proc." Roy. Soc., May, 1915) it was shown by the author that the quantity of electronic radiation liberated from an atom by a beam of X-radiation is directly proportional to the fourth-power of the weight of the atom—*i.e.*, if two elemental gases, having equal numbers of atoms per unit volume, are subjected to identical beams of X-rays, the numbers of electrons liberated in these gases per unit length of X-ray beam are proportional to the fourth powers of their atomic weights. The liberation of electrons in a gas by the passage of an X-ray beam through it is accompanied by an absorption of the X rays in the gas. The absorption of the X-ray beam in a given length of a gas is proportional to the number of electrons liberated by the X-rays in traversing this portion of its path. This being so, it is evident that the coefficients of absorption of a given type of X-rays in two elemental gases must be proportional to the fourth power of their atomic weights, provided the number of atoms per unit volume is the same in the two gases.

* Barkla and Simons, "Phil. Mag.," Feb., 1912; C. T. R. Wilson, Roy. Soc. "Proc.," June, 1912; Barkla and Philpot, "Phil. Mag.," June, 1913.

† H. Moore, "Phil. Mag.," Jan., 1914.

The liberation of electrons in a compound gas by X-rays being an atomic phenomenon, the absorption of X-rays in such a gas must also be atomic—*i.e.*, the absorption in a compound is the sum of the absorptions that would be produced, if the various constituent atoms were present in the same numbers but not in combination. This atomic property of the absorption in compounds was shown by actual experiment by Bencist as early as 1901.

The present Paper shows how, for any type of homogeneous X-radiation, the absorption coefficient in a hypothetical, monatomic vapour of any element at a pressure of 76 cm. can be calculated, if the coefficient of absorption of these rays is known in air, and provided that the radiations excited in the hypothetical vapour belong to the same sets of series as those excited in air by the same incident beam (*see* pp. 436 and 437). From the absorption coefficient calculated for this hypothetical vapour the absorption coefficient for the same type of X-rays can be calculated for the same element in any condition, a simple density law holding good whether the element is solid, liquid, or gaseous. By an additive law, the absorption coefficient of any compound can be calculated from the absorptions of the various constituent elements, and thus it should be possible to calculate eventually the coefficient of absorption of any substance of known constitution for any type of X-radiation whose absorption coefficient is known in air, provided the radiations excited in all these cases belong to the same sets of series (*see* pp. 436 and 437). It is shown in the Paper to which reference has been made* that if a monatomic vapour of an element at a pressure of 76 cm. could be subjected to a homogeneous beam of X-rays from copper, the number of electrons liberated in unit length of the vapour could be expressed in the form $\frac{(\text{atomic weight})^4}{\text{a constant}}$. If the number of electrons liberated by the same beam of X-rays in passing through unit length of air is taken as unity, the mean value of this constant was found to be $1{\cdot}05 \times 10^5$. This constant is based on the *ratio* between the electronic radiations liberated in the vapour and in air, and should, therefore, have the same value for all types of X-radiation, provided the result of the absorption is similar in all cases (*see also* pp. 436 and 437). Calculations based on this assumption show a fairly good agreement with observed values for radiations

* Moore, "Proc." Roy. Soc., May, 1915.

other than copper, thus bearing out this conclusion. It has, of course, been abundantly shown that the absorption coefficients of two materials bear a constant ratio for all X-radiations, provided the types of secondary radiation excited in the two materials are similar.

A beam of X-radiation passing through unit length of one of these hypothetical vapours would cause the liberation of a quantity of electrons which can be calculated in terms of the quantity emitted in unit length of the same X-ray beam in air, the aperture being supposed to be the same in both cases. The absorption coefficients for this hypothetical vapour and for air would bear the same ratio to each other as these quantities of electronic radiation, and thus, if the absorption coefficient in air is known for this X-ray beam, the absorption coefficient in the vapour can be calculated. The density of the vapour is calculated from its atomic weight.

Absorption in Aluminium.

Aluminium is taken as an example of the method of calculation, because the absorption coefficients of different radiation are usually determined in thicknesses of aluminium.

A monatomic vapour of aluminium at 76 cm. would have an absorption coefficient for any type of radiation, equal to $\frac{(27)^4}{105{,}000}$ times that of air for the same radiation—*i.e.*, for monatomic aluminium vapour at 76 cm.

$$\lambda = 5{\cdot}06 \times \lambda_{air}.$$

The absorption coefficients of this vapour for different types of radiation have been calculated on this assumption, and will be found in column 3, Table I. These numbers are based on the absorption coefficients found experimentally for the same radiation in air.

The density of the vapour would be 0·00121 grm. per cubic centimetre, and the density of solid aluminium is 2·7 grm. per cubic centimetre; applying the density law, the actual coefficient of absorption in aluminium sheet will be 2,230 times the absorption coefficient for this hypothetical vapour, or

$$\lambda_{Al} = 2{,}230 \times 5{\cdot}06 \times \lambda_{air}.$$

Values for λ_{Al}, calculated for different radiations by this method, are given in column 4, Table I., and in column 5 of the same table are the coefficients actually determined by various experimenters.

TABLE I.—*Absorptions in Aluminium.*

Radiator.	λ in air (observed).	λ in hypothetical aluminium vapour (calculated).	λ in aluminium sheet (calculated).	λ in aluminium sheet (observed).
Iron	0·0201	0·1017	227	239
Copper	0·0109	0·055	123	128
Zinc	0·00898	0·0455	102	106
Bromine	0·00389	0·0197	43·9	44·0
Silver.....	0·00076(2)	0·0038(5)	8·5	6·75
Tin.....	0·00050(3)	0·0025(4)	5·6	4·24

NOTE.—The observed values are taken or deduced from the values given in Kaye's book on "X-Rays."

The agreement between the observed and calculated values is quite close for the iron, copper, zinc and bromine radiations, but is less exact for the harder radiations of silver and tin. Even in these cases, the difference is not more than 25 per cent., and is probably within the limits of accuracy obtainable in the majority of the experiments from which the observed values are quoted, at any rate for the harder rays. The determination of absorption coefficients is rendered extremely difficult by the enormous scattering correction which has to be applied, this correction being greater for the harder rays than for rays of longer wave-length. The scattering coefficient is only approximately known, and this could easily lead to errors of 25 per cent. or over, in the values obtained for the absorption coefficients of the harder rays.

Absorptions in Other Elements.

If different radiations are supposed to be absorbed in various elemental substances, and values for the coefficients of absorption are calculated as for aluminium, a sudden disagreement occurs in some cases between the calculated and observed values as the atomic weight of the absorber becomes greater than that of the radiator.

This is shown in the following table :—

TABLE II.—*Absorptions in Various Metals.*

Absorber.	λ (calculated) for radiations from					λ (observed) for radiations from				
	Fe.	Cu.	Br.	Ag.	Sn.	Fe.	Cu.	Br.	Ag.	Sn.
Mg	107	57·3	20·6	4·03	2·67	139	72	...	3·82	...
Al.....	227	123	43·9	8·5	5·6	239	128	44	6·75	4·24
Fe	630	330	112	21·9	14·5	520	21,000	...	137	...
Ni	765	413	148	29·0	19·1	746	458	...	22	...
Cu	986	531	191	37·5	24·7	850	552	...	21·7	...
Ag	5,600	3,300	1,085	213·5	141	4,000	225	...	139	...

The values for magnesium and aluminium give approximate agreement, wherever observed values are available for comparison with the calculated values. The calculated and observed absorptions in iron, however, though of the same order of magnitude for iron radiation, are entirely different for the radiations from metals of higher atomic weight, the calculated values being much greater than the observed.

Nickel and copper, on the other hand, show an approximate agreement throughout the whole range of radiations considered, notwithstanding the fact that the atomic weights of the radiators are some above and some below the atomic weights of the absorbers.

Silver is different again, giving calculated and observed values of the same order of magnitude for radiation from iron, but the values for all the other radiations are of entirely different orders.

These anomalies are connected with the selective absorption of X-radiation in the different metals, and are probably due to the fact that in some cases the homogeneous radiation corresponding with the K series is absent (*see also* p. 437).

Absorption in Compounds.

With compounds, the absorption coefficients calculated on the additive law by similar methods agree fairly well with the observed values where known, provided the constituent elements have atomic weights below that of the radiator.

As an example of this, the value was calculated for the absorption coefficients of SO_2 for different radiations. The coefficients for monatomic vapours of sulphur and oxygen at 76 cm. were first calculated and the value for SO_2 determined by addition; the results obtained are given in Table III. A similar set of values calculated for ethyl bromide gave no agreement whatever with the observed values, bromine having an atomic weight greater than that of the radiator used, the K radiation from bromine being, therefore, not excited (*see* p. 437).

TABLE III.—*Absorptions in* SO_2.

Absorber.	λ for radiations from				
	Fe.	Cu.	Br.	Ag.	Sn.
Monatomic sulphur vapour at 76 cm. (hypothetical).	0·201	0·1085	0·0389	0·00762	0·00503
Monatomic oxygen at 76cm. (hypothetical)	0·0125	0·00672	0·00241	0·00047	0·00031
SO_2 (calculated from above)	0·226	0·122	0·0437	0·0085	0·00565
SO_2 (observed)*	0·24	0·134	0·050	0·0079	...

* Barkla and Collier, "Phil. Mag.," June, 1912.

The sudden disagreement between the calculated and observed values of the absorption coefficients, which occurs as the atomic weight of the absorber becomes greater than that of the radiator, is very marked in the case of iron, though it is hardly shown at all with copper and nickel. It would appear that this discontinuity is associated with selective absorption.

If a radiation of definite wave-length, say, the K radiation from some element, is allowed to fall on materials of atomic weight less than that of the radiator, the absorption would produce characteristic radiations from the absorber belonging to all the series K, L, M, &c. When the atomic weight of the absorber is greater than that of the radiator, the K radiation from the absorber is no longer excited, and the absorption produces the L, M, &c., radiations only. The sudden cessation of the K radiation might be expected to produce a discontinuity in the relation between the absorption coefficient and the atomic weight of the absorber, and this would correspond with the change noticed.

It might be supposed that the atomic absorption for a given radiation is an additive quantity.

For the K radiation excited in the absorber, the absorption may be some quantity Ax^4, where A is a constant, and x the atomic weight of the absorber. The L radiation excited demands a further absorption Bx^4, the M and other radiations (if any) requiring absorptions Cx^4, Dx^4, &c. In each case a fourth-power law holds good, but a discontinuity occurs when one of the radiations ceases to be excited, the absorption coefficients being given by $(A+B+C+D+..)x^4$ when all the radiations are given off, and changing to $(B+C+D+..)x^4$ when the K radiation ceases.

Prof. Bragg and Mr. Pierce * have shown that the atomic absorption coefficient does vary in some such way. A fourth-power law is obeyed fairly accurately up to a certain value of the atomic weight, but at this point a discontinuity occurs. For absorbers of higher atomic weight the atomic absorption is still proportional to the fourth power of the atomic weight or of the atomic number, but the constant coefficient is different.

In their calculations, the atomic absorptions were compared with the fourth power of the atomic numbers of the absorbers. The accuracy of the absorption coefficients determined is not,

* "Phil. Mag.," Oct., 1914.

however, sufficiently great to determine whether the atomic number or the atomic weight should be used.

There is no doubt that the absorption of X-radiation requires much further investigation before it is at all completely understood, and the present Paper is only offered as a possible solution of one part of the phenomena connected therewith. In any futher investigations it will be essential to ensure strict homogeneity of the X-ray beam used, and every attempt should be made to obtain a higher order of accuracy than has been attained in most of the experiments up to the present.

ABSTRACT.

The action of X radiation when passing through a gas is to liberate electrons from the gas. The number of electrons emitted by any atom in a beam of X-rays is proportional to the fourth-power of its atomic weight (or possibly its atomic number). (Moore, "Proc." Roy. Soc., May, 1915.) Thus, equal numbers of atoms of different elements, when subjected to similar X-ray beams, will liberate amounts of electronic radiation proportional to the fourth powers of the atomic weights of the elements.

The absorption coefficients are proportional to the amounts of electronic radiation liberated, and, therefore, the absorptions of two elements, when equal numbers of atoms are present, will be proportional to the fourth powers of their atomic weights.

The corpuscular radiation liberated in the vapour of an element if it could be obtained as a monatomic vapour at 76 cm. can be expressed as $1{\cdot}05 \times 10^5 \times (\text{atomic weight})^4$, taking the corpuscular radiation in air as unity. The absorption coefficient of such a vapour would, therefore, be this number of times the coefficient of absorption of air for the same type of X-radiation. The absorption of any element is proportional to the number of atoms present, and having calculated the absorption in a hypothetical vapour of this type, the absorption in the same element in any condition can be calculated by a simple density law.

This is done in the Paper for several elements (metals), and also, assuming an additive law, it has been calculated for some compounds. The agreement between the calculated values and the values obtained by different observers by direct experiment is quite close over a considerable range of radiations and absorbers. When, however, the atomic weight of the absorber is higher than that of the radiator, so that the K series is absent from any secondary radiation excited in the absorber, the agreement ceases. A possible explanation of this is suggested in the Paper.

DISCUSSION.

Prof. O. W. Richardson thought the Paper made a considerable addition to our knowledge of the process of ionisation by X-rays. The primary rays in being absorbed produce some ionisation, and, in addition, excite secondary radiation, which in turn produces more ions. The Paper helps to make clear the part played by these two causes in the production of the total ionisation.

Dr. S. Russ felt that the Paper conveyed the impression that one would find in practice the same value for the absorption coefficient as obtained by calculation. The phenomenon of selective absorption must surely set severe limits to the range of wave-length over which these relations hold. This range should be specified. How would the author calculate the absorption coefficients, say, for aluminium and water? If there were no selective absorption λ/ρ should be constant, but he had recently found it to vary over a wide range.

Prof. Nicholson said the results were of interest to those engaged in the attempt to construct models of atoms and molecules. The phenomena appeared to be atomic in nature, thus bearing out the idea brought forward by Bragg that the atom preserves its identity in the molecule. In connection with the law connecting the absorption with the fourth power of the atomic weight, it would be interesting to know whether it was really the atomic weight or the atomic number that should be employed.

The Author replied as follows: The formula should only be used where all the types of radiation excited in air are also excited in the material. He had had considerable trouble in getting reliable coefficients owing to the large and uncertain scattering corrections which had to be applied. In one case he believed an observed value of 0·42 had to be corrected to 0·2 or less. He had calculated some coefficients for gases, and with SO_2 the agreement was as good as for solids. It should be possible to calculate the true absorption coefficient for water and aluminium, but he could not say how near these would be to values actually determined on account of the uncertain scattering. He had experiments in progress which should determine whether the atomic weight or the atomic number was the significant quantity.

At the meeting of the Society held on Friday, May 28, 1915, at University of London, King's College, the following demonstrations and exhibitions were given :—

Two Experiments Illustrating Novel Properties of the Electron Currents from Hot Metals were shown by Prof. O. W. RICHARDSON, *M.A., D.Sc., F.R.S.*

THE first experiment demonstrates the cooling of a tungsten filament when an electron current is allowed to flow from its surface. This effect is analogous to the cooling due to latent heat when a liquid evaporates, or to the similar phenomenon due to the heat of reaction when a gas is emitted by chemical decomposition of a solid.

An experimental lamp containing a fine filament of double tungsten is placed in one arm of a balanced Wheatstones bridge actuated by the current which heats the wire. When the electron current is allowed to flow, by completing a side circuit from an electrode inside the lamp to a point in the adjacent arm of the bridge, the galvanometer is deflected in a direction which corresponds to a reduction of the resistance (and temperature) of the hot filament. The precautions necessary fully to eliminate disturbances due to various subsidiary phenomena are considered in the following Papers by H. L. Cooke and the writer ("Phil. Mag.," Vol. XXV., p. 624, 1913, and *ibid.* Vol. XXVI., p. 472, 1913).

The second experiment, in which a similar experimental lamp is used, demonstrates the flow of electron currents from a hot filament to a surrounding cylinder *against* various opposing potential differences up to about 1 volt. On account of the large currents from tungsten this effect can easily be shown on a galvanometer. The data can be used to find the velocities of the emitted electrons (cf. "Phil. Mag.," Vol. XVI., p. 353; Vol. XVII., p. 890; Vol. XVIII., p. 681).

On High Permeability in Iron.

PROF. E. WILSON gave a short account of the experiments, and exhibited the apparatus, which he had recently described in Papers read before the Royal Society. In these was dis-

cussed the increase of permeability of silicon iron produced by heat treatment with application of magnetic fields during cooling, and by demagnetisation while screened from the earth's magnetic field.

An Experiment Showing the Difference in Width of the Spectrum Lines of Neon and Hydrogen was shown by Mr. T. R. MERTON.

FOR accurate measurements of the widths of spectrum lines an interference method must be employed, in which the optical difference of path between the inferfering beams can be varied, the widths being calculated from the limiting difference of path at which interference fringes can be seen. By "crossing" a Fabry & Perot etalon with a single prism spectroscope it is possible to discriminate between lines arising from different elements, by the "visibility of the fringes." In the experiment shown a vacuum tube containing neon and hydrogen is examined in this way. The neon lines being narrow, show sharp interference fringes, but for the hydrogen lines, which are broader, the limiting order at which interference can be seen is too low for fringes to be visible.

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